ADVANCED

ENGINEERING MATHEMATICS

SIXTH EDITION

Chapter 7

Lecture1-Vector

Asst.Prof.Dr. Santhad Chuwongin

Outline

1. Vectors in 2-Space (2D)
2. Vectors in 3-Space (3D)
3. Dot Product
4. Cross Product
5. Lines and Planes in 3-Space
6. Vector Spaces
7. Gram–Schmidt Orthogonalization Process

เว็ปไซท์ : www.symbolab.com

Vectors in 2-Space

- A **scalar** is a real number or quantity that has a magnitude, such as length and temperature
- A **vector** has both magnitude and direction and it can be represented by a boldface symbol or a symbol under an arrow, **v** or $\overrightarrow{AB}$

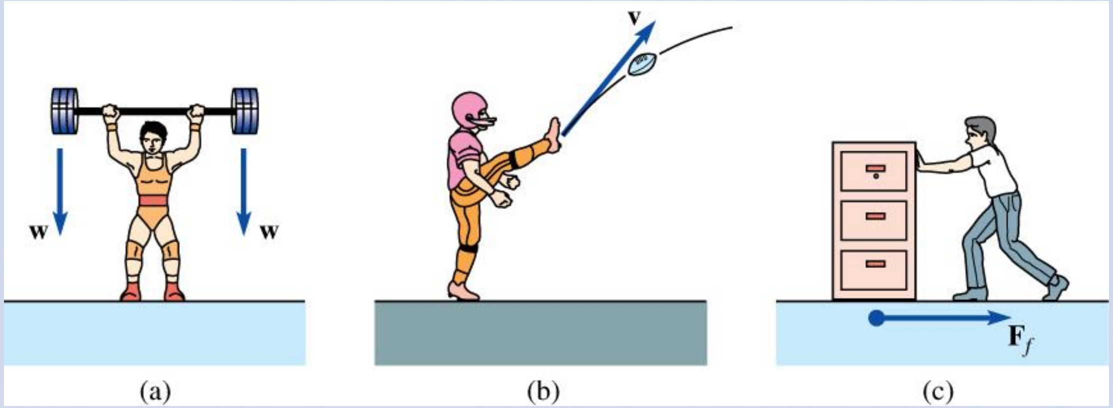


Figure 07.1.1: Examples of vector quantities

Vectors in 2-Space (cont'd.)

- Characteristics of vectors
  - The vector $\overrightarrow{AB}$ has an initial point at A and a terminal point at B
  - **Equal** vectors have the same magnitude and direction
  - The **negative** of a vector has the same magnitude and opposite direction

Vectors in 2-Space (cont'd.)

- Characteristics of vectors (cont'd.)
  - If $k \neq 0$ is a scalar, the **scalar multiple** of a vector $k\overrightarrow{AB}$ is a vector that is $|k|$ times $\overrightarrow{AB}$
  - Two vectors are **parallel** if they are nonzero scalar multiples of each other

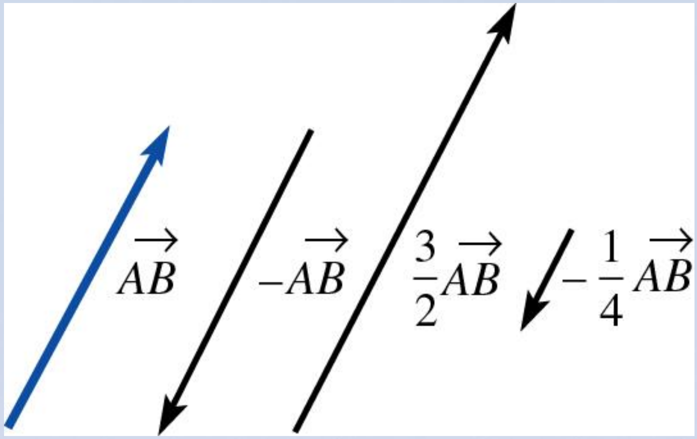


Figure 07.1.3: Parallel vectors

Addition and Subtraction of 2-space vectors

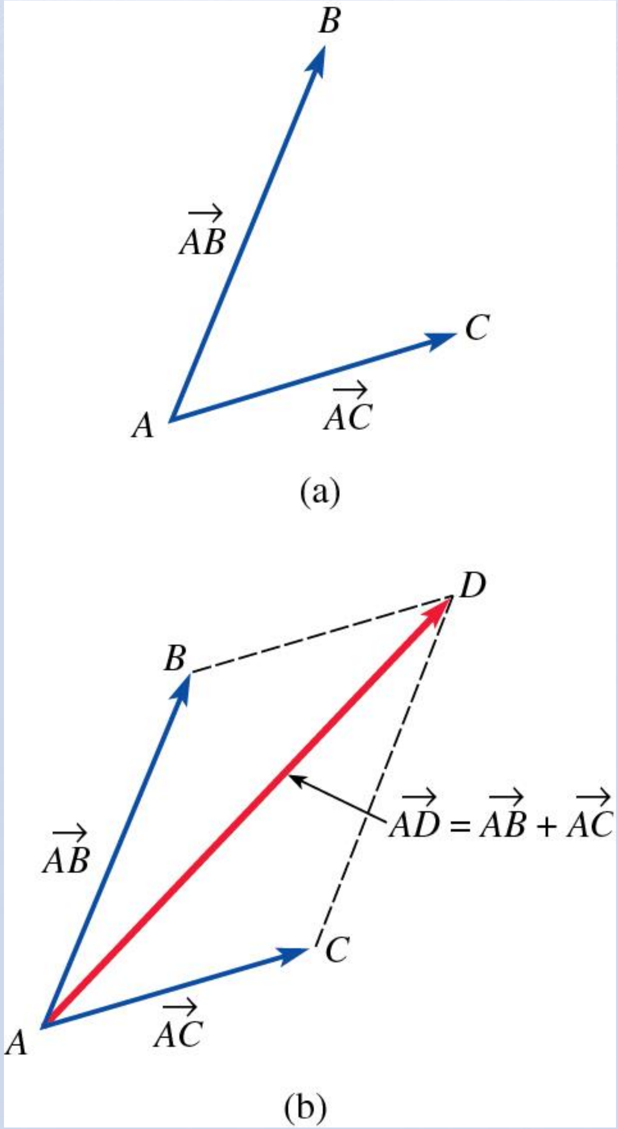


Figure 07.1.4: Vector $\overrightarrow{AD}$ is the sum of $\overrightarrow{AB}$ and $\overrightarrow{AC}$

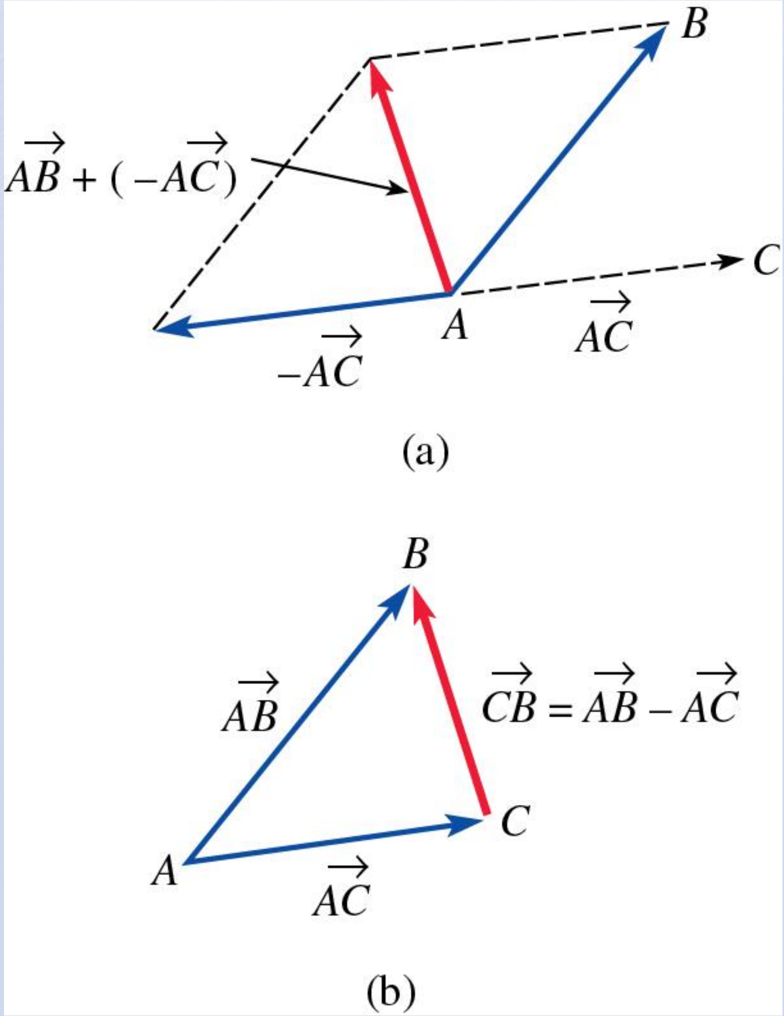
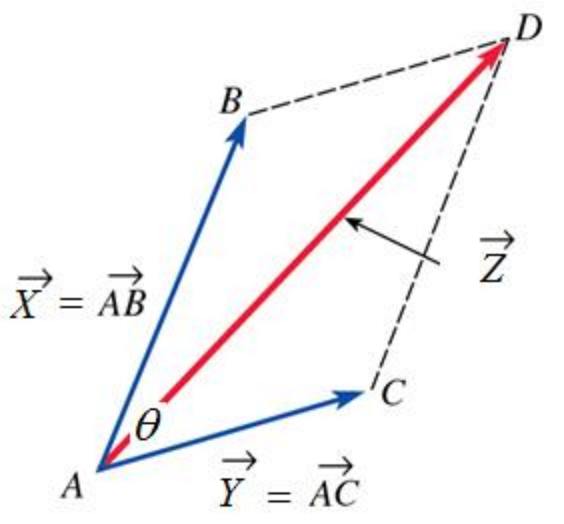


Figure 07.1.5: Vector $\overrightarrow{CB}$ is the difference of $\overrightarrow{AB}$ and $\overrightarrow{AC}$

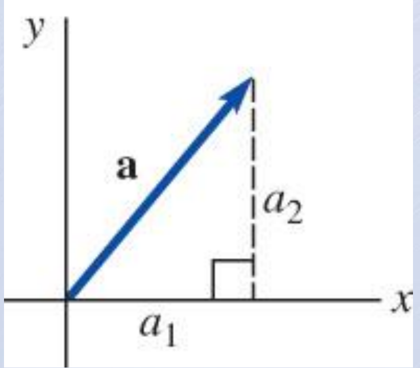
Addition of 2-space vectors



Law of Cosines

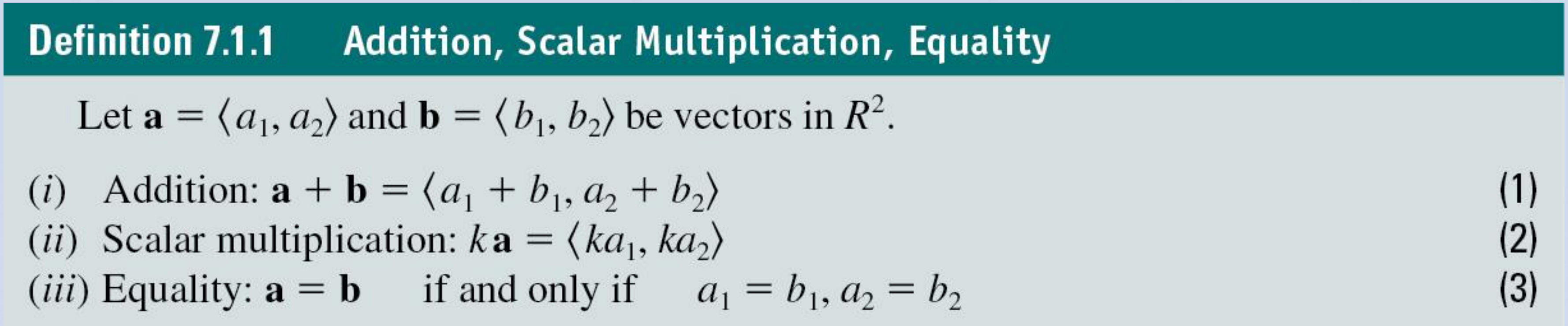
$$\left|\vec{Z}\right| = \sqrt{\left|\vec{X}\right|^2 + \left|\vec{Y}\right|^2 + 2\left|\vec{X}\right|\left|\vec{Y}\right|\cos(\theta)}$$

Vectors in 2-Space (cont'd.)



$$\mathrm{a} = 2\mathrm{i} - 3\mathrm{j} = \langle 2, -3 \rangle$$

- A vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is an ordered pair of real numbers where a_1 and a_2 are the **components** of the vector
  - Addition and subtraction of vectors, multiplication of vectors by scalars, and so on, are defined in terms of components



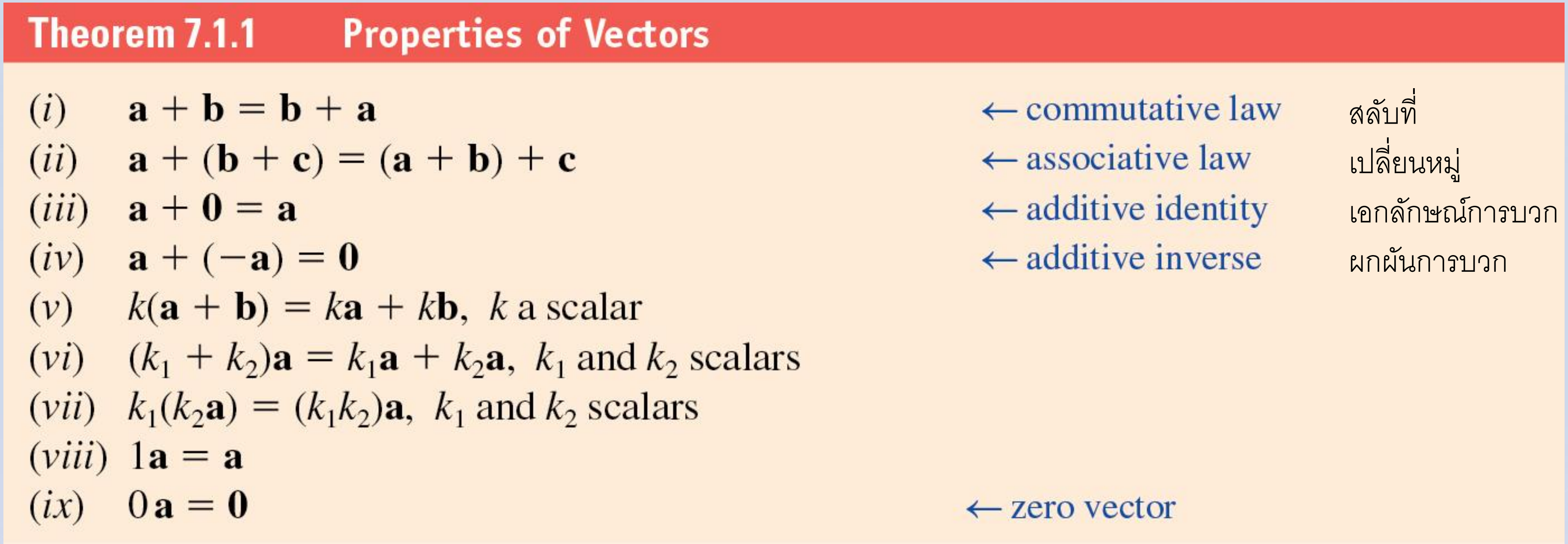
Definition 7.1.1 Addition, Scalar Multiplication, Equality

Let $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ be vectors in R^2.

(*i*) Addition: $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$ (1)
(*ii*) Scalar multiplication: $k\mathbf{a} = \langle ka_1, ka_2 \rangle$ (2)
(*iii*) Equality: $\mathbf{a} = \mathbf{b}$ if and only if $a_1 = b_1, a_2 = b_2$ (3)

Vectors in 2-Space (cont'd.)

- The component definition of a vector can be used to verify the following properties of vectors



Theorem 7.1.1 Properties of Vectors

(*i*) $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ ← commutative law สลับที่

(*ii*) $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$ ← associative law เปลี่ยนหมู่

(*iii*) $\mathbf{a} + \mathbf{0} = \mathbf{a}$ ← additive identity เอกลักษณ์การบวก

(*iv*) $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$ ← additive inverse ผกผันการบวก

(*v*) $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$, k a scalar

(*vi*) $(k_1 + k_2)\mathbf{a} = k_1\mathbf{a} + k_2\mathbf{a}$, k_1 and k_2 scalars

(*vii*) $k_1(k_2\mathbf{a}) = (k_1k_2)\mathbf{a}$, k_1 and k_2 scalars

(*viii*) $1\mathbf{a} = \mathbf{a}$

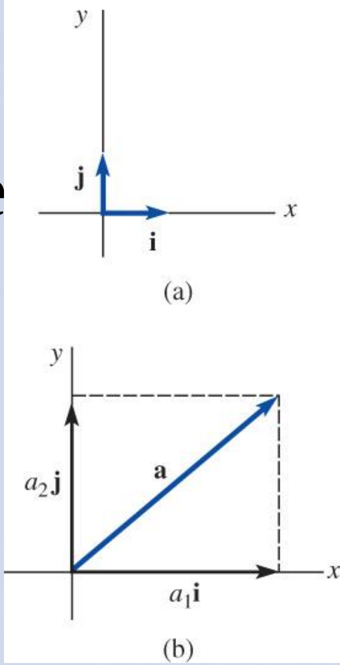
(*ix*) $0\mathbf{a} = \mathbf{0}$ ← zero vector

Vectors in 2-Space (cont'd.)

- The **magnitude**, **length**, or **norm** of a vector **a** is denoted by $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2}$
- A vector **u** with magnitude 1 is a **unit vector**
  - $\mathbf{u} = \left(1/\|\mathbf{a}\|\right)\mathbf{a}$ is the normalization of **a**
  - The unit vectors $\mathbf{i} = \langle 1,0 \rangle$ and $\mathbf{j} = \langle 0,1 \rangle$ are the standard basis for two-dimensional vectors

$$\mathbf{a} = \langle a_1, a_2 \rangle = a_1\mathbf{i} + a_2\mathbf{j}$$

where a_1 and a_2 are **horizontal** and **vertical** **components** of **a**, respectively



Vectors in 2-Space (cont'd.)

EXAMPLE Vector Operations Using **i** and **j**

(a) $\langle 4, 7\rangle = 4\mathbf{i} + 7\mathbf{j}$

(b) $(2\mathbf{i} - 5\mathbf{j}) + (8\mathbf{i} + 13\mathbf{j}) = 10\mathbf{i} + 8\mathbf{j}$

(c) $\|\mathbf{i} + \mathbf{j}\| = \sqrt{2}$

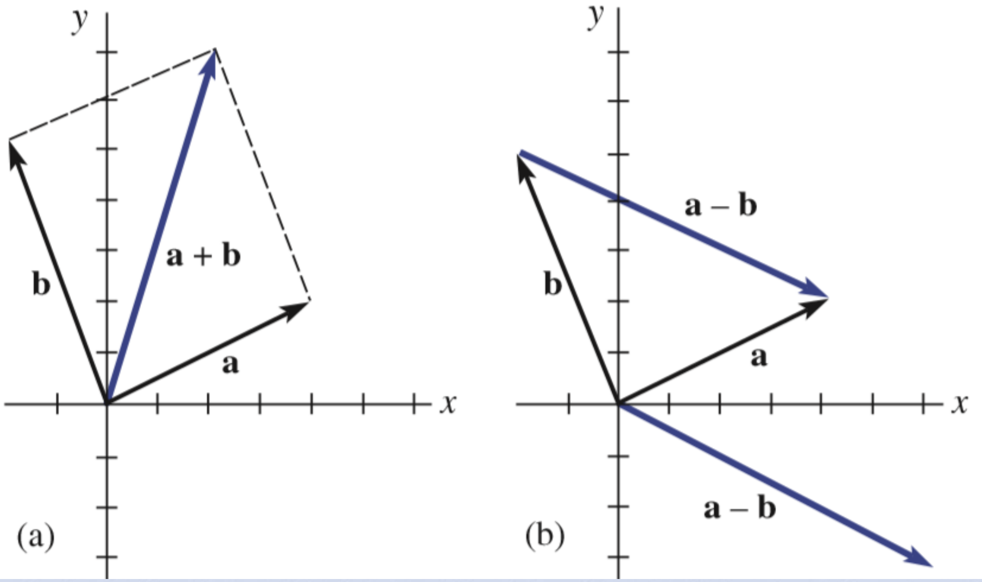
(d) $10(3\mathbf{i} - \mathbf{j}) = 30\mathbf{i} - 10\mathbf{j}$

(e) $\mathbf{a} = 6\mathbf{i} + 4\mathbf{j}$, $\mathbf{b} = 9\mathbf{i} + 6\mathbf{j}$ are parallel, since **b** is a scalar multiple of **a**. $\mathbf{b} = \frac{3}{2}\mathbf{a}$.

EXAMPLE Graphs of Vector Sum/Vector Difference

Let $\mathbf{a} = 4\mathbf{i} + 2\mathbf{j}$ and $\mathbf{b} = -2\mathbf{i} + 5\mathbf{j}$. Graph $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$.

SOLUTION The graphs of $\mathbf{a} + \mathbf{b} = 2\mathbf{i} + 7\mathbf{j}$ and $\mathbf{a} - \mathbf{b} = 6\mathbf{i} - 3\mathbf{j}$ are given



Vectors in 3-Space

- In three dimensions, or **3-space**, a rectangular coordinate system is constructed with three mutually orthogonal axes

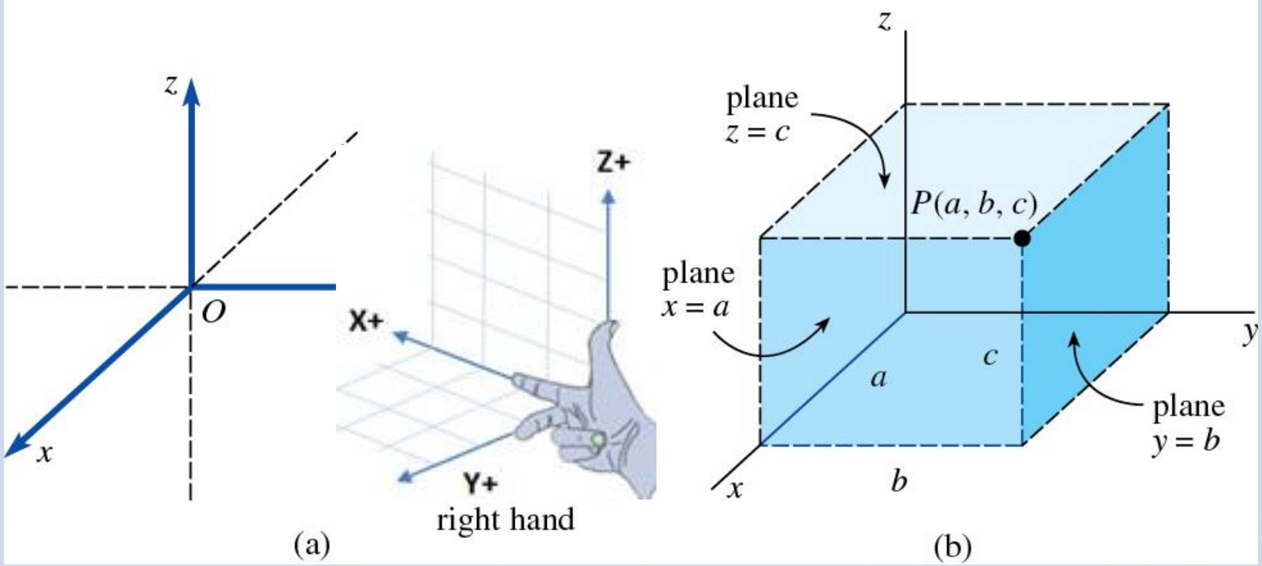


Figure 07.2.2: Rectangular coordinates in 3-space

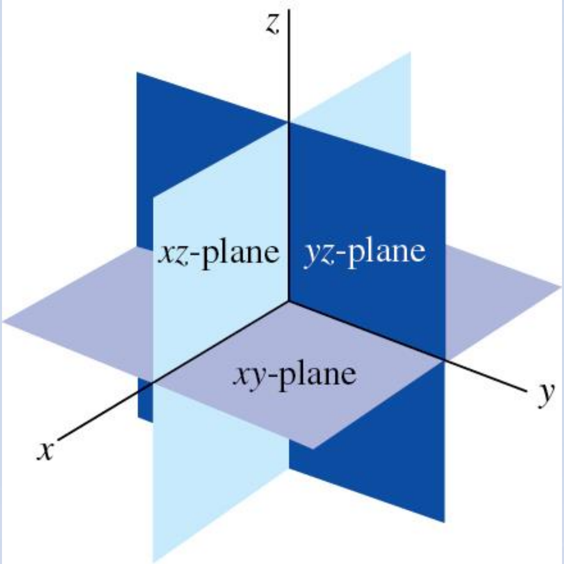


Figure 07.2.3: Octants

Vectors in 3-Space (cont'd.)

- The **distance** between two points

$$d\left(P_1,P_2\right)=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2+\left(z_2-z_1\right)^2}$$

จงหาระยะทาง (d) ระหว่างจุด (2,-3,6) และ (-1,-7,4)

$$\mathrm{d}=\sqrt{(2-(-1))^2+(-3-(-7))^2+(6-4)^2}=\sqrt{9+16+4}=\sqrt{29}$$

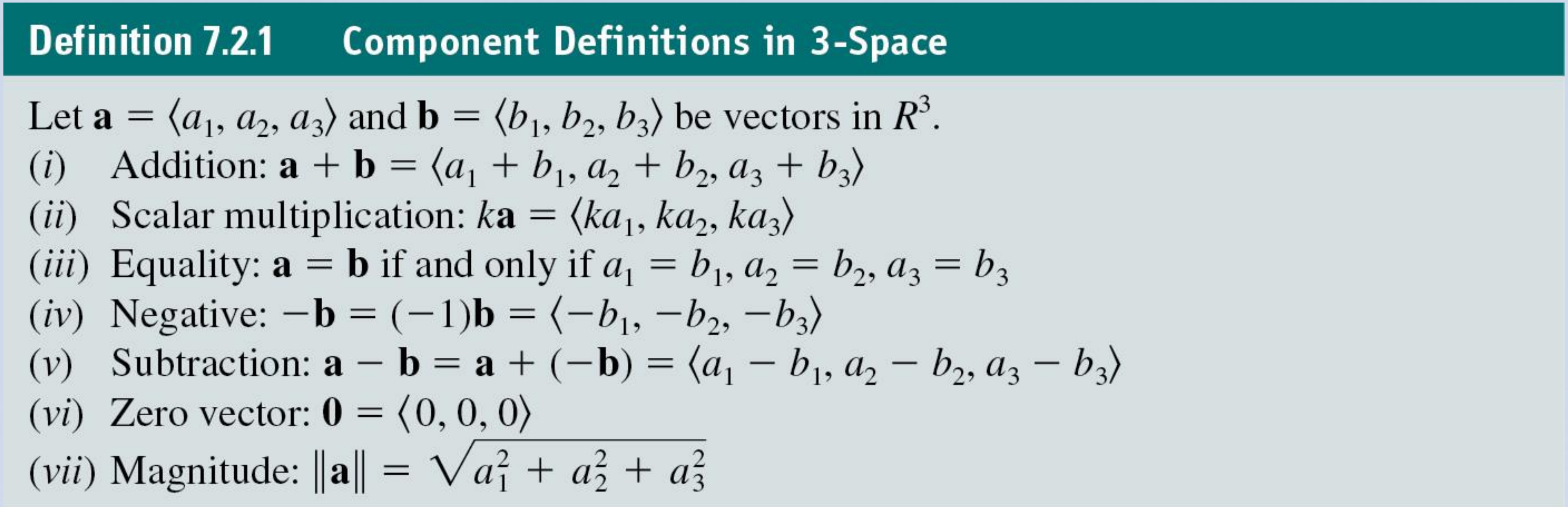
- The coordinates of the **midpoint** of a line segment between two points

$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$$

Vectors in 3-Space (cont'd.)

- A vector **a** in 3-space is any ordered triple of real numbers

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle$$



Definition 7.2.1 Component Definitions in 3-Space

Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ be vectors in R^3.

(*i*) Addition: $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

(*ii*) Scalar multiplication: $k\mathbf{a} = \langle ka_1, ka_2, ka_3 \rangle$

(*iii*) Equality: $\mathbf{a} = \mathbf{b}$ if and only if $a_1 = b_1, a_2 = b_2, a_3 = b_3$

(*iv*) Negative: $-\mathbf{b} = (-1)\mathbf{b} = \langle -b_1, -b_2, -b_3 \rangle$

(*v*) Subtraction: $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$

(*vi*) Zero vector: $\mathbf{0} = \langle 0, 0, 0 \rangle$

(*vii*) Magnitude: $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Vectors in 3-Space (cont'd.)

- Any vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ can be expressed as a linear combination of the unit vectors

$$\mathbf{i} = \langle 1, 0, 0 \rangle \qquad \mathbf{j} = \langle 0, 1, 0 \rangle \qquad \mathbf{k} = \langle 0, 0, 1 \rangle$$

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

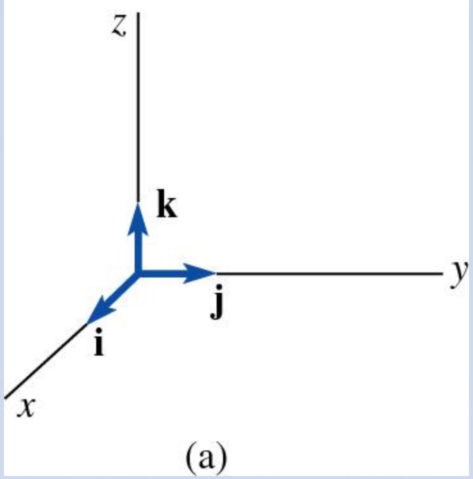
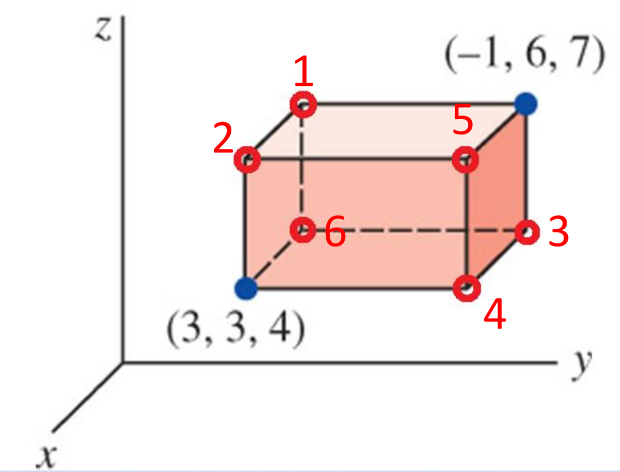


Figure 07.2.8a: **i**, **j**, and **k** form a basis for R^3



จุด 1= (-1,3,7)

จุด 2= (3,3,7)

จุด 3= (-1,6,4)

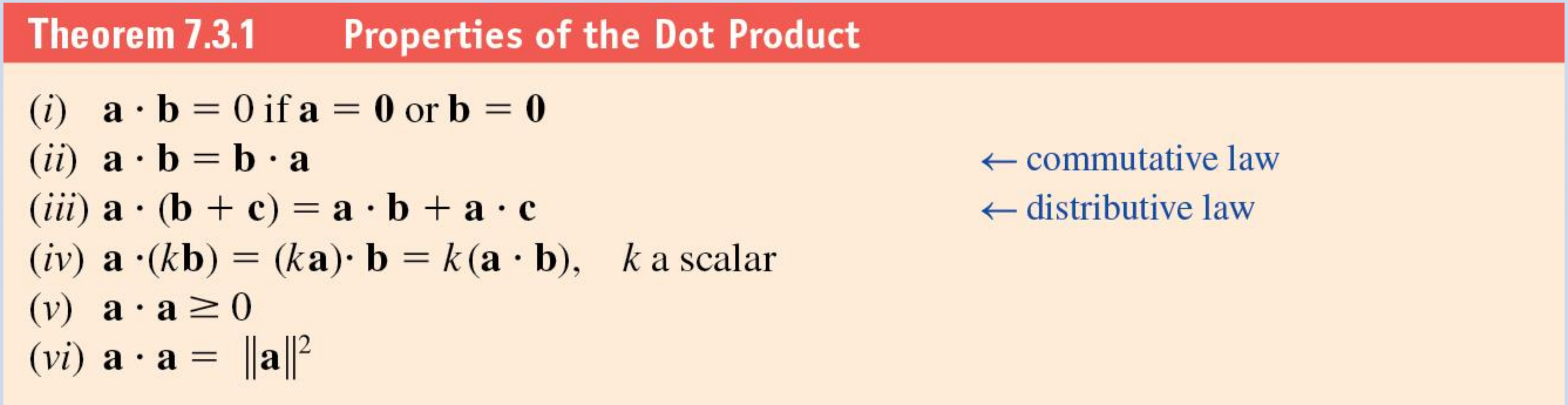
จุด 4= (3,6,4)

จุด 5,6 = ?

(3,6,7), (-1,3,4)

Dot Product

- In 2-space, $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2$
- In 3-space,
- In n-space,



Theorem 7.3.1 Properties of the Dot Product

(*i*) $\mathbf{a} \cdot \mathbf{b} = 0$ if $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$

(*ii*) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ ← commutative law

(*iii*) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ ← distributive law

(*iv*) $\mathbf{a} \cdot (k\mathbf{b}) = (k\mathbf{a}) \cdot \mathbf{b} = k(\mathbf{a} \cdot \mathbf{b})$, k a scalar

(*v*) $\mathbf{a} \cdot \mathbf{a} \geq 0$

(*vi*) $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$

- Alternative form of the dot product $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos\theta$

$\mathrm{a} = -3\mathrm{i} - \mathrm{j} + 4\mathrm{k}, \mathrm{b} = 2\mathrm{i} + 14\mathrm{j} + 5\mathrm{k}, \quad \mathrm{a} \cdot \mathrm{b} =$

Dot Product (cont'd.)

- 2 vectors (**a** and **b**) are *orthogonal* if **a·b = 0**
- The angle between two vectors is given by

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\|\mathbf{a}\|\|\mathbf{b}\|}$$

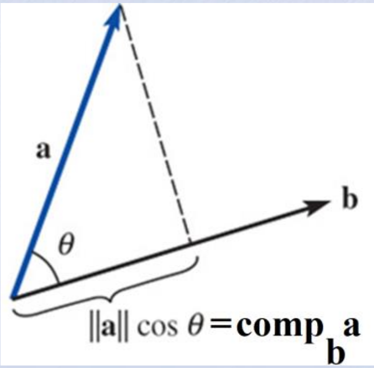
$$\mathrm{a} = 2\mathrm{i} + 3\mathrm{j} + \mathrm{k}, \mathrm{b} = -\mathrm{i} + 5\mathrm{j} + \mathrm{k}$$

$$\|\mathrm{a}\| = \sqrt{14}, \|\mathrm{b}\| = \sqrt{27}, \mathrm{a.b} = 14$$

$$\cos\theta = \frac{14}{\sqrt{14}\sqrt{27}} \Rightarrow \theta \approx 43.9°$$

Component of a on b (ปริมาณสเกลาร์)

- The component of **a** on **b** คือ $\text{comp}_{\mathbf{b}}\mathbf{a} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|}$



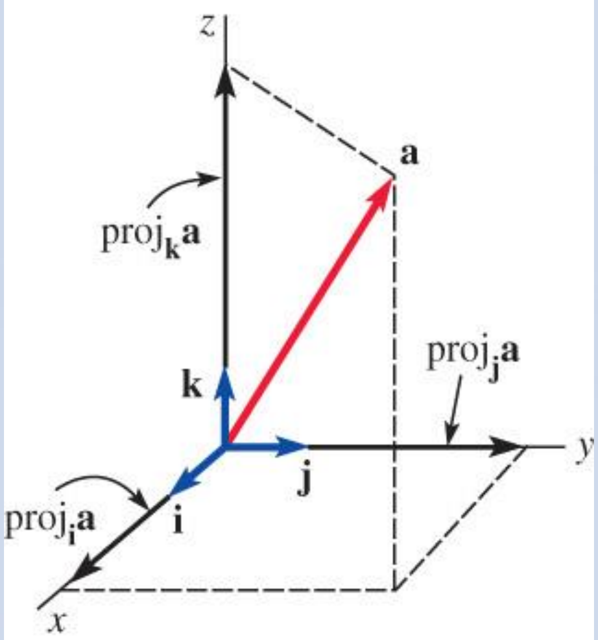
Component of a on b

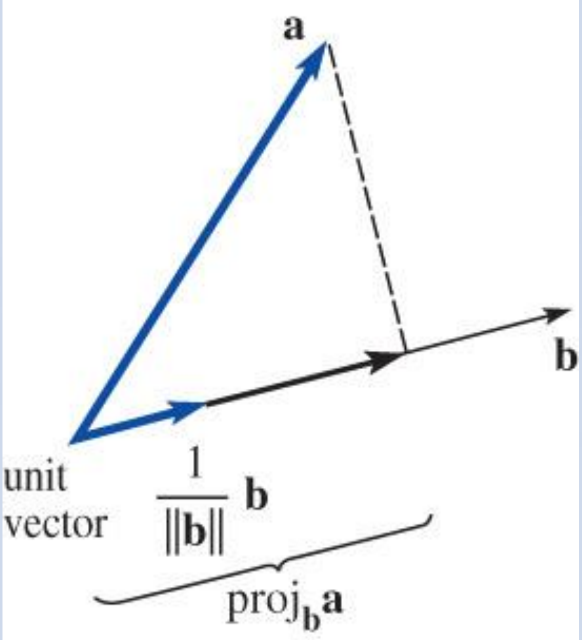
- The component of **a** on **i** $\Rightarrow \text{comp}_{\text{i}}\text{a} = \dfrac{\text{a} \cdot \text{i}}{\|\text{i}\|} = \text{a} \cdot \text{i}$
- The component of **a** on **j** $\Rightarrow \text{comp}_{\text{j}}\text{a} = \dfrac{\text{a} \cdot \text{j}}{\|\text{j}\|} = \text{a} \cdot \text{j}$
- The component of **a** on **k** $\Rightarrow \text{comp}_{\text{k}}\text{a} = \dfrac{\text{a} \cdot \text{k}}{\|\text{k}\|} = \text{a} \cdot \text{k}$

Component of **a** on a vector **b**, we dot **a** with a unit vector in the direction of **b**

Projection of **a** on **b** (ปริมาณเว็คเตอร์)

$$\mathrm{proj_b a} = (\mathrm{comp_b a})\left(\overbrace{\frac{\mathrm{b}}{\|\mathrm{b}\|}}^{\text{เว็คเตอร์หน่วยตามแนว b}}\right) = \left(\frac{\mathrm{a \cdot b}}{\|\mathrm{b}\|}\right)\left(\frac{\mathrm{b}}{\|\mathrm{b}\|}\right) = \left(\frac{\mathrm{a \cdot b}}{\mathrm{b \cdot b}}\right)\mathrm{b}$$





Projection of **a** on **b**

$$\mathrm{proj}_{\mathrm{b}}\mathrm{a} = (\mathrm{comp}_{\mathrm{b}}\mathrm{a})\left(\frac{\mathrm{b}}{\|\mathrm{b}\|}\right) = \left(\frac{\mathrm{a}\cdot\mathrm{b}}{\|\mathrm{b}\|}\right)\left(\frac{\mathrm{b}}{\|\mathrm{b}\|}\right) = \left(\frac{\mathrm{a}\cdot\mathrm{b}}{\mathrm{b}\cdot\mathrm{b}}\right)\mathrm{b}$$

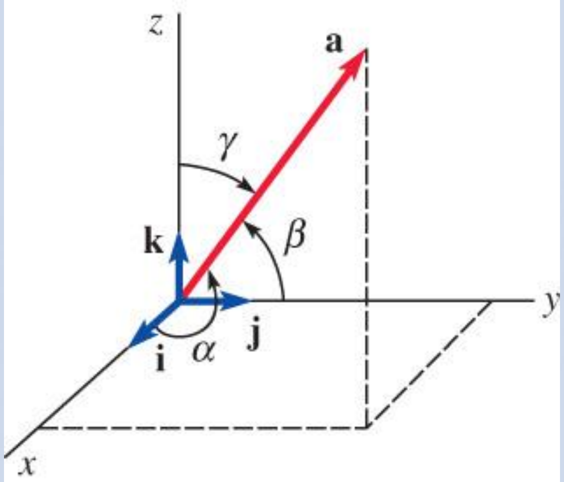
Find the projection of $\mathbf{a} = 4\mathbf{i} + \mathbf{j}$ onto the vector $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j}$. Graph.

Dot Product (Direction Cosines)

ถ้า $\mathrm{a} = \mathrm{a}_1\mathrm{i} + \mathrm{a}_2\mathrm{j} + \mathrm{a}_3\mathrm{k}$, มุม α, β, γ ระหว่างเวคเตอร์ a และเวคเตอร์ i, j, k จะถูกเรียกว่า *“Direction Angles”*

$$\cos\alpha = \frac{\mathrm{a}\cdot\mathrm{i}}{\|\mathrm{a}\|\|\mathrm{i}\|}, \cos\beta = \frac{\mathrm{a}\cdot\mathrm{j}}{\|\mathrm{a}\|\|\mathrm{j}\|}, \cos\gamma = \frac{\mathrm{a}\cdot\mathrm{k}}{\|\mathrm{a}\|\|\mathrm{k}\|}$$

$$\cos\alpha = \frac{\mathrm{a}_1}{\|\mathrm{a}\|}, \cos\beta = \frac{\mathrm{a}_2}{\|\mathrm{a}\|}, \cos\gamma = \frac{\mathrm{a}_3}{\|\mathrm{a}\|}$$



$\cos\alpha, \cos\beta, \cos\gamma$ are called "*Direction Cosines*"

$$\mathrm{a} = 2\mathrm{i} + 5\mathrm{j} + 4\mathrm{k}$$

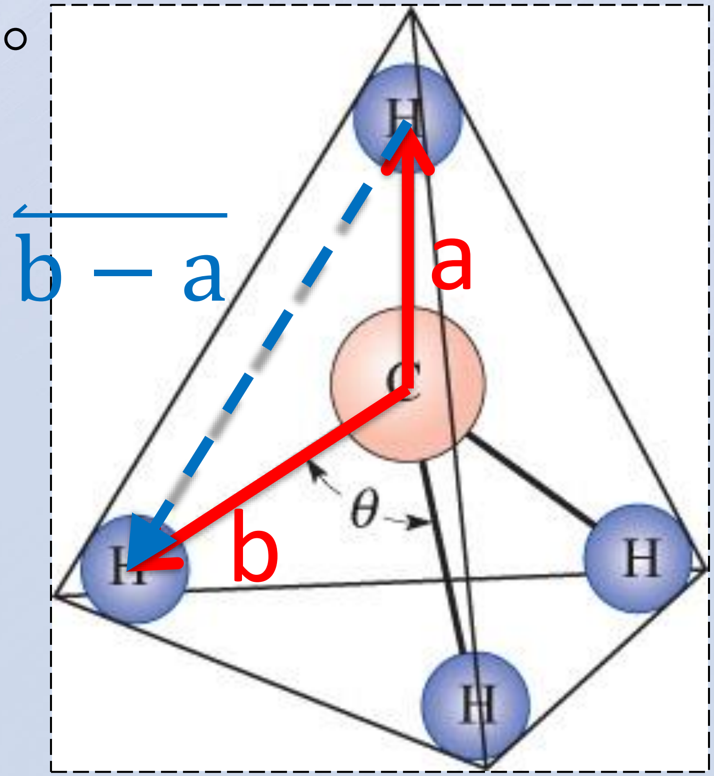
$$\alpha = \cos^{-1}\left(\frac{2}{\sqrt{45}}\right) \approx 72.7^\circ \qquad \beta = \cos^{-1}\left(\frac{5}{\sqrt{45}}\right) \approx 41.8^\circ \qquad \gamma = \cos^{-1}\left(\frac{4}{\sqrt{45}}\right) \approx 53.4^\circ$$

ตัวอย่าง

สมมุติให้ระยะ C − H เท่ากับ 1.1 Å ,θ=109.5°

จงหาระยะ H − H=?

ระยะ H − H = $\|\mathbf{b} - \mathbf{a}\|$



$$\|\mathbf{b}-\mathbf{a}\| = \sqrt{(\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a})} = \sqrt{\mathbf{b}\cdot\mathbf{b} - 2\mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{a}}$$

Cross Product

- **Determinants** (ทบทวน)

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \qquad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

- **Cross Product** of 2 vectors is given by

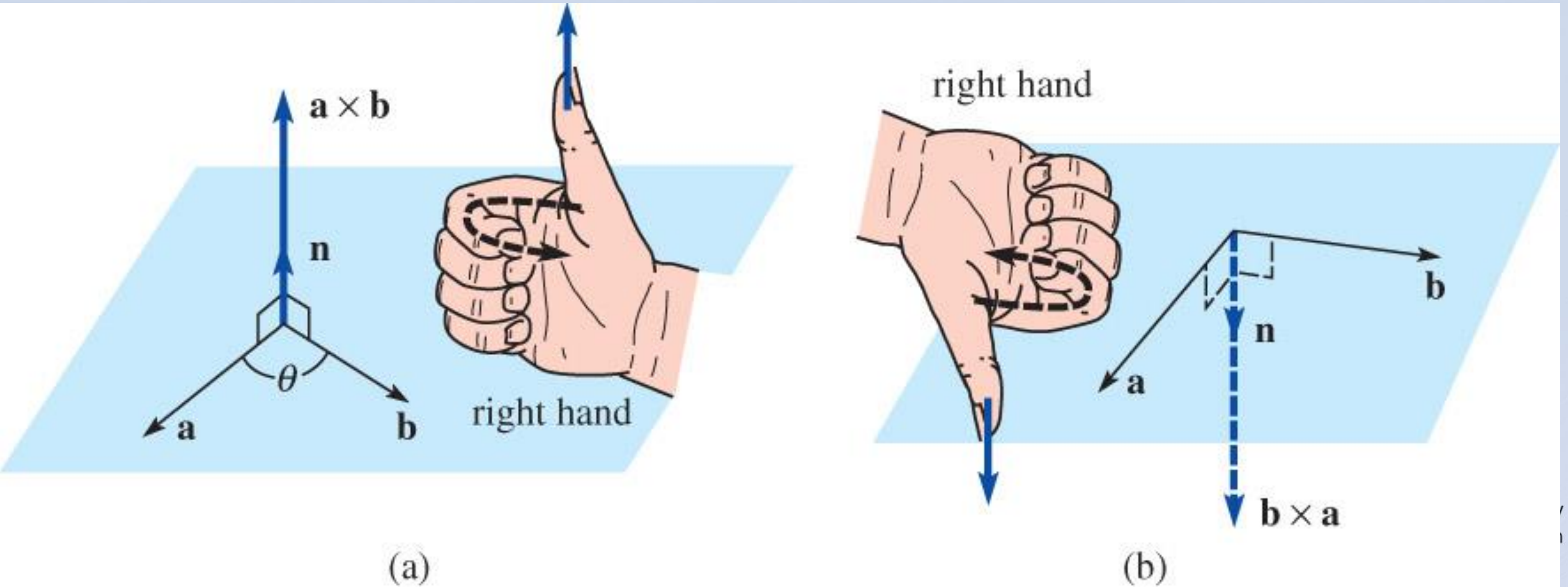
$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \left(a_2b_3 - a_3b_2\right)\mathbf{i} - \left(a_1b_3 - a_3b_1\right)\mathbf{j} + \left(a_1b_2 - a_2b_1\right)\mathbf{k}$$

Cross Product (cont'd.)

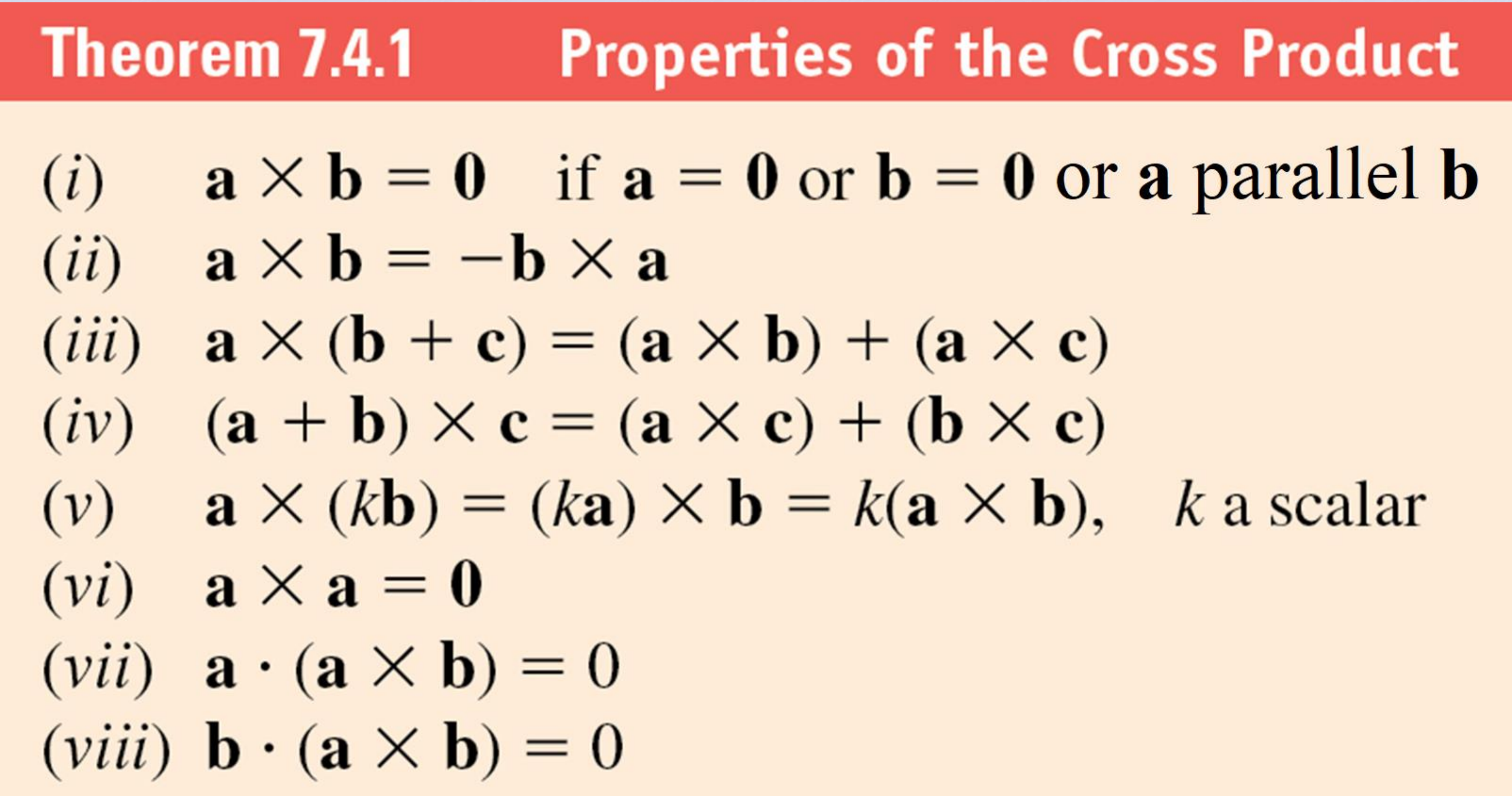
- $\mathbf{a}\times\mathbf{b}$ is *orthogonal* to the plane containing **a** and **b**
- The magnitude of the cross product is given by

$$\left\|\mathbf{a}\times\mathbf{b}\right\| = \left\|\mathbf{a}\right\|\left\|\mathbf{b}\right\|\sin\theta$$

- 2 nonzero vectors are parallel if $\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\|\,\mathbf{sin\,0°} = \mathbf{0}$
- $\mathbf{a}\cdot\left(\mathbf{b}\times\mathbf{c}\right)=0$ if **a**, **b**, and **c** are coplanar



Cross Product (cont'd.)



Theorem 7.4.1 Properties of the Cross Product

(*i*) $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ if $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ or $\mathbf{a}$ parallel $\mathbf{b}$

(*ii*) $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

(*iii*) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$

(*iv*) $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c})$

(*v*) $\mathbf{a} \times (k\mathbf{b}) = (k\mathbf{a}) \times \mathbf{b} = k(\mathbf{a} \times \mathbf{b})$, k a scalar

(*vi*) $\mathbf{a} \times \mathbf{a} = \mathbf{0}$

(*vii*) $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$

(*viii*) $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$

Lines and Planes in 3-Space

- The **vector equation** for a line is $\mathbf{r} = \mathbf{r}_2 + t\mathbf{a}$
  - $\mathbf{r}$, $\mathbf{r}_1$, $\mathbf{r}_2$ คือเว็คเตอร์จากจุดกำเนิดชี้ไปที่เส้นตรง
  - สเกลาร์ t คือพารามิเตอร์
  - เว็คเตอร์ $\mathbf{a}$ คือเว็คเตอร์ทิศทาง (มีทิศทางเดียวกับเว็คเตอร์ $\mathbf{r}$-$\mathbf{r}_2$)

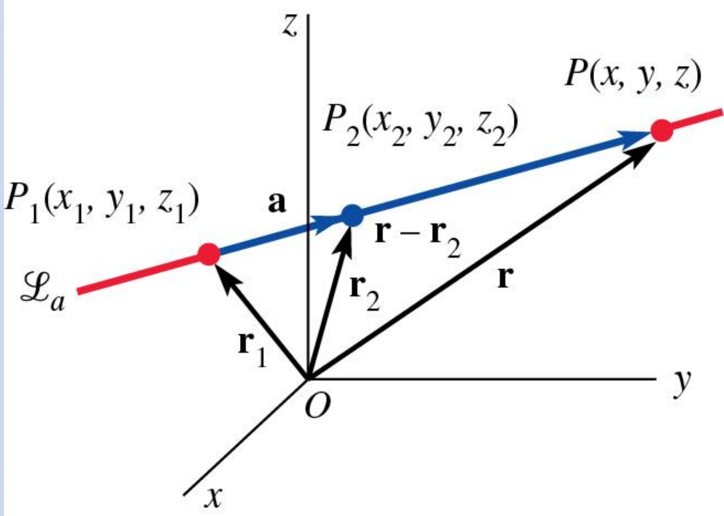
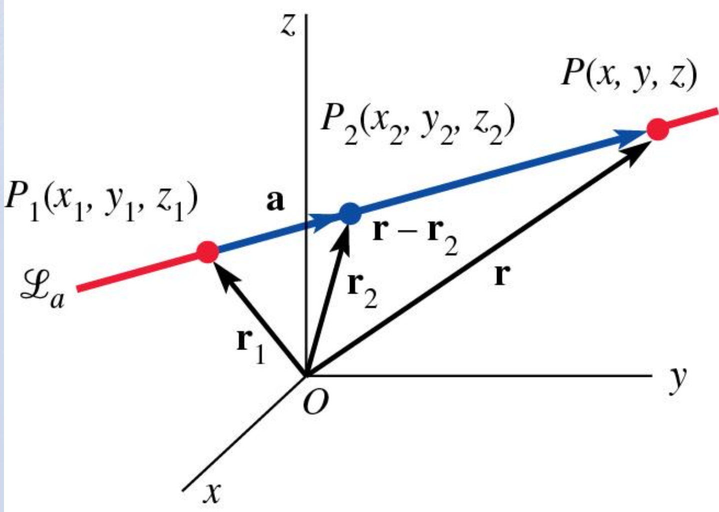


Figure 07.5.1: Line through distinct points in 3-space

Example



- จงหาสมการเวคเตอร์สำหรับเส้นตรงที่ผ่าน (2,-1,8) และ (5,6,-3)

a = <2-5,-1-6,8-(-3)> = <-3,-7,11>

หรือ

a = <5-2,6-(-1),-3-8> = <3,7,-11>

ดังนั้น สมการเวคเตอร์ มีค่าเท่ากับ

$<x,y,z> = r_2 + t*a = <2,-1,8> + t<-3,-7,11>$

$<x,y,z> = r_2 + t*a = <5,6,-3> + t<3,7,-11>$

Lines and Planes in 3-Space (cont'd.)

- The **vector equation** for a plane is $\mathbf{n}\cdot\left(\mathbf{r}-\mathbf{r}_1\right)=0$
  - Plane passes through a given point and has a specified normal vector **n**
  - $\mathbf{r}_1$ and **r** are vectors from the origin (0,0) to points P_1, P on the plane

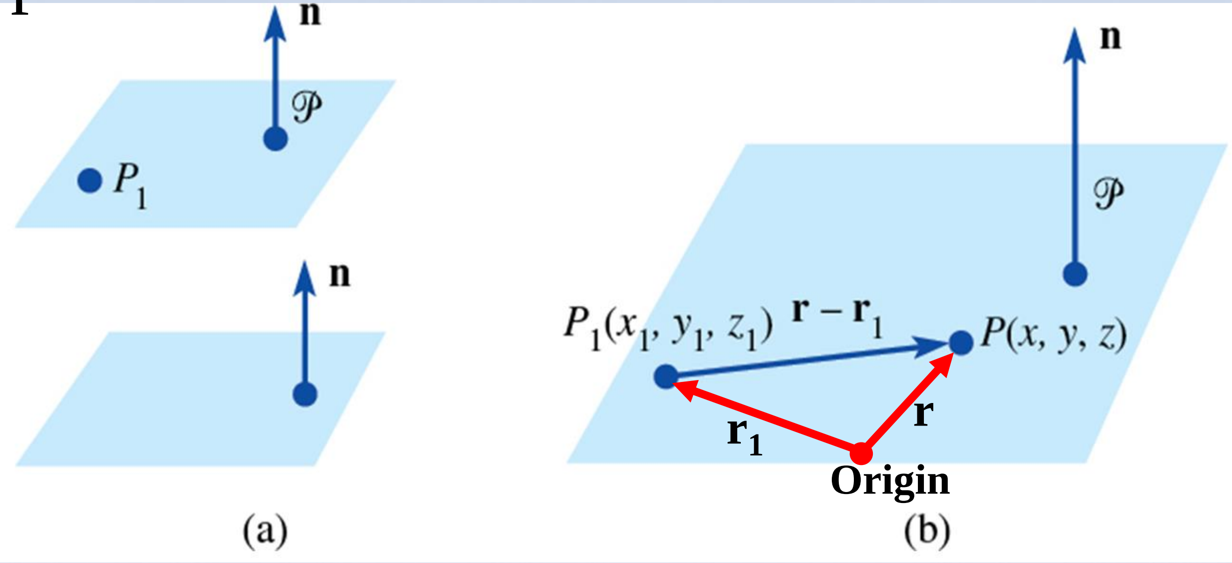
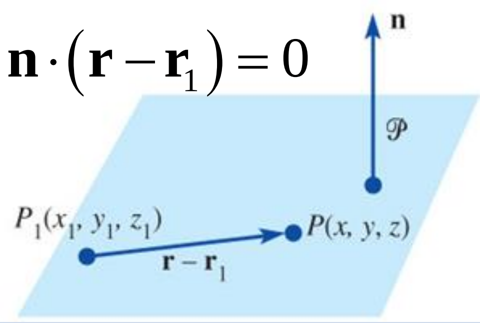


Figure 07.5.3: Vector **n** is perpendicular to a plane

Ascend Learning Company
www.jblearning.com

Examples

1. จงหาสมการของระนาบซึ่งมีเวคเตอร์ตั้งฉาก $n = 2\mathbf{i} + 8\mathbf{j} - 5\mathbf{k}$ และมีจุด $(4, -1, 3)$ อยู่บนระนาบดังกล่าว?



$$\overrightarrow{P_1P} = r - r_1 = (x-4)\mathbf{i} + (y+1)\mathbf{j} - (z-3)\mathbf{k} = 0$$

$$n.(r - r_1) = 2(x-4) + 8(y+1) - 5(z-3) = 0$$

$\therefore$ สมการของระนาบ คือ $2x + 8y - 5z + 15 = 0$

2. จงหาสมการของระนาบที่มีจุด $(1,0,-1), (3,1,4)$ และ $(2,-2,0)$ อยู่บนระนาบ

$$\left.\begin{matrix}(3,1,4)\\(1,0,-1)\end{matrix}\right\} u = 2\mathbf{i} + 1\mathbf{j} + 5\mathbf{k},$$

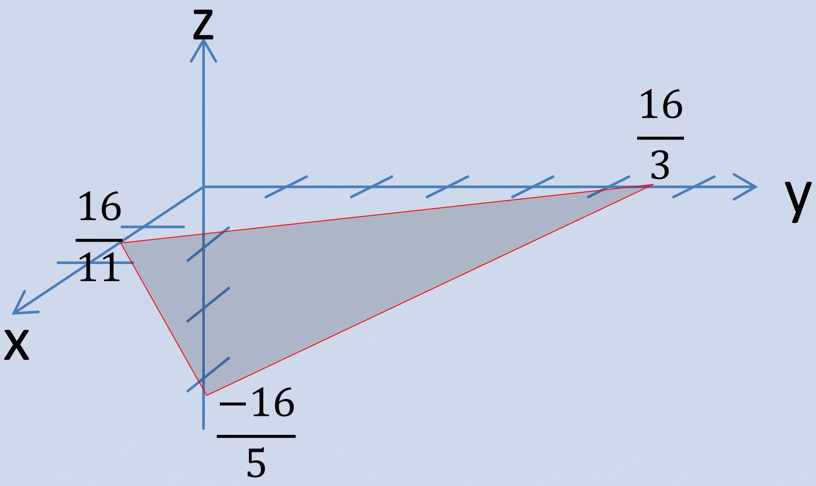
$$\left.\begin{matrix}(3,1,4)\\(2,-2,0)\end{matrix}\right\} v = 1\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$$

$$\left.\begin{matrix}(x,y,z)\\(2,-2,0)\end{matrix}\right\} w = (x-2)\mathbf{i} + (y+2)\mathbf{j} + z\mathbf{k}$$

$$u \times v = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 1 & 5\\ 1 & 3 & 4\end{vmatrix} = -11\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$$

$$(u \times v).w = 0 = -11(x-2) - 3(y+2) + 5z$$

$$-11x - 3y + 5z + 16 = 0$$



Examples

3. จงหาสมการของระนาบที่มีจุด (1, 2, -1), (4, 3, 1), (6, 4, 4) อยู่บนระนาบ

$$\left.\begin{matrix}(1,2,-1)\\(4,3,1)\end{matrix}\right\} \mathrm{u} = 3\mathbf{i} + 1\mathbf{j} + 2\mathbf{k}$$

$$\left.\begin{matrix}(4,3,1)\\(6,4,4)\end{matrix}\right\} \mathrm{v} = 2\mathbf{i} + 1\mathbf{j} + 3\mathbf{k}$$

$$\left.\begin{matrix}(4,3,1)\\(x,y,z)\end{matrix}\right\} \mathrm{w} = (x-4)\mathbf{i} + (y-3)\mathbf{j} + (z-1)\mathbf{k}$$

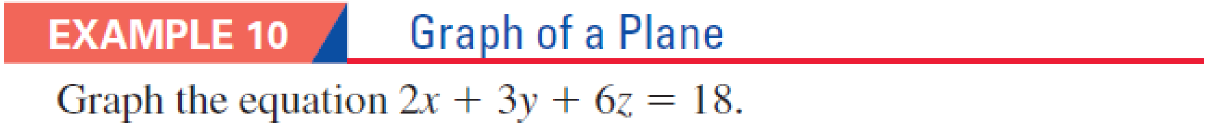
$$\mathrm{u} \times \mathrm{v} = \begin{vmatrix} i & j & k \\ 3 & 1 & 2 \\ 2 & 1 & 3 \end{vmatrix} = \mathbf{i} - 5\mathbf{j} + \mathbf{k}$$

$$(\mathrm{u} \times \mathrm{v}).\mathrm{w} = 0 = (x-4) - 5(y-3) + (z-1)$$

$$x - 5y + z + 10 = 0$$

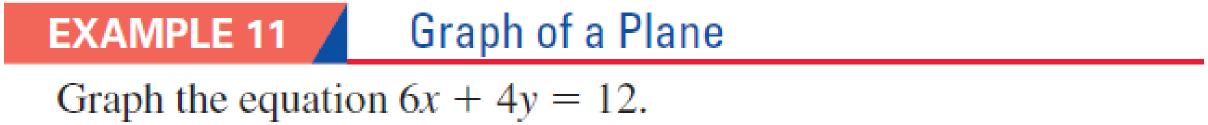
$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_1) = 0$$

Examples



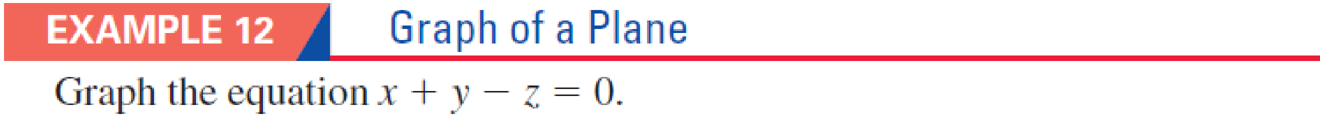
EXAMPLE 10 Graph of a Plane

Graph the equation $2x + 3y + 6z = 18$.



EXAMPLE 11 Graph of a Plane

Graph the equation $6x + 4y = 12$.



EXAMPLE 12 Graph of a Plane

Graph the equation $x + y - z = 0$.

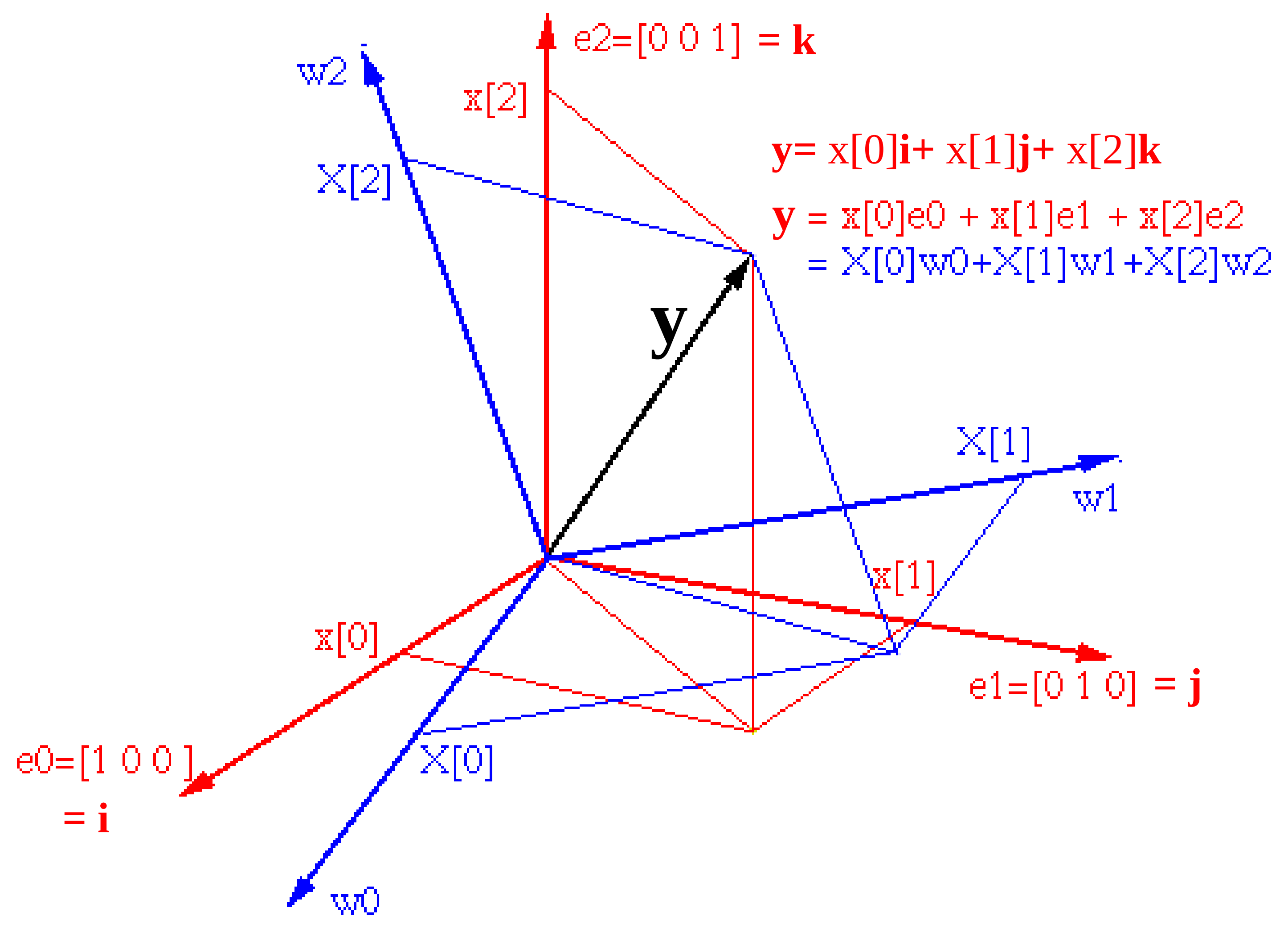
Orthonormal Basis (ฐานเชิงตั้งฉาก)

- Every vector **u** in R^2 (n=2) can be written as a linear combination of the vectors in the standard basis $B = \{\mathbf{e_1}, \mathbf{e_2}\}$, where $\mathbf{e_1} = \hat{\boldsymbol{x}} = \mathbf{i} = \langle 1,0 \rangle$, $\mathbf{e_2} = \hat{\boldsymbol{y}} = \mathbf{j} = \langle 0,1 \rangle$
- Every vector **u** in R^3 can be written as a linear combination of the vectors in the standard basis $B = \{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\} = \{\mathrm{i}, \mathrm{j}, \mathrm{k}\}$ หรือ $\mathrm{i} = \langle 1,0,0 \rangle$, $\mathrm{j} = \langle 0,1,0 \rangle$, $\mathrm{k} = \langle 0,0,1 \rangle$

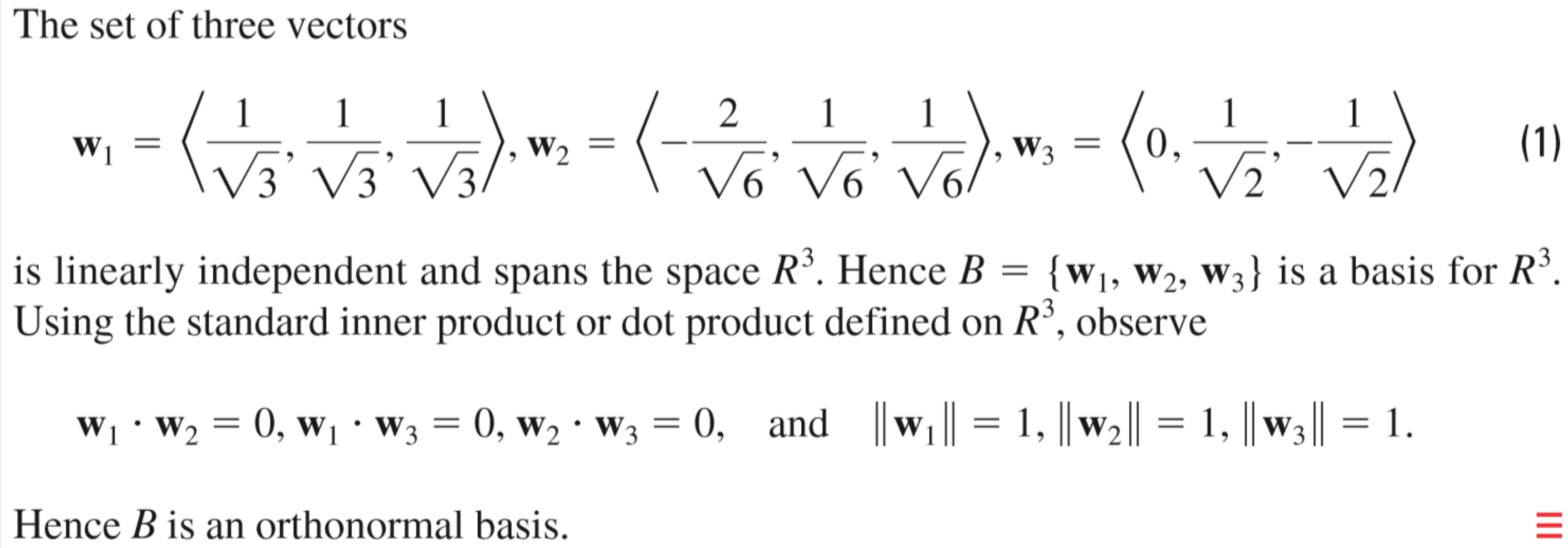
Orthonormal Basis (ฐานเชิงตั้งฉาก)

- Every vector $\mathbf{u}$ in R^n can be written as a linear combination of the vectors in the standard basis $B = \{\mathbf{e_1}, \mathbf{e_2}, \dots, \mathbf{e_n}\}$, where $\mathbf{e_1} = \langle 1,0,0,\dots,0\rangle$, $\mathbf{e_2} = \langle 0,1,0,\dots,0\rangle, \dots,$ $\mathbf{e_n} = \langle 0,0,0,\dots,1\rangle$
- Orthonormal basis = mutually orthogonal ($\mathbf{e}_\mathrm{i} \cdot \mathbf{e}_\mathrm{j} = 0, \mathrm{i} \neq \mathrm{j}$) and unit vectors ($\|\mathbf{e}_\mathrm{i}\| = 1$, $\mathrm{i} = 1,2,\dots,n$)
- HOW TO transform or convert any basis B of R^n into an orthonormal basis?

Example of Orthonormal Basis for R^3



Example of Orthonormal Basis for R^n



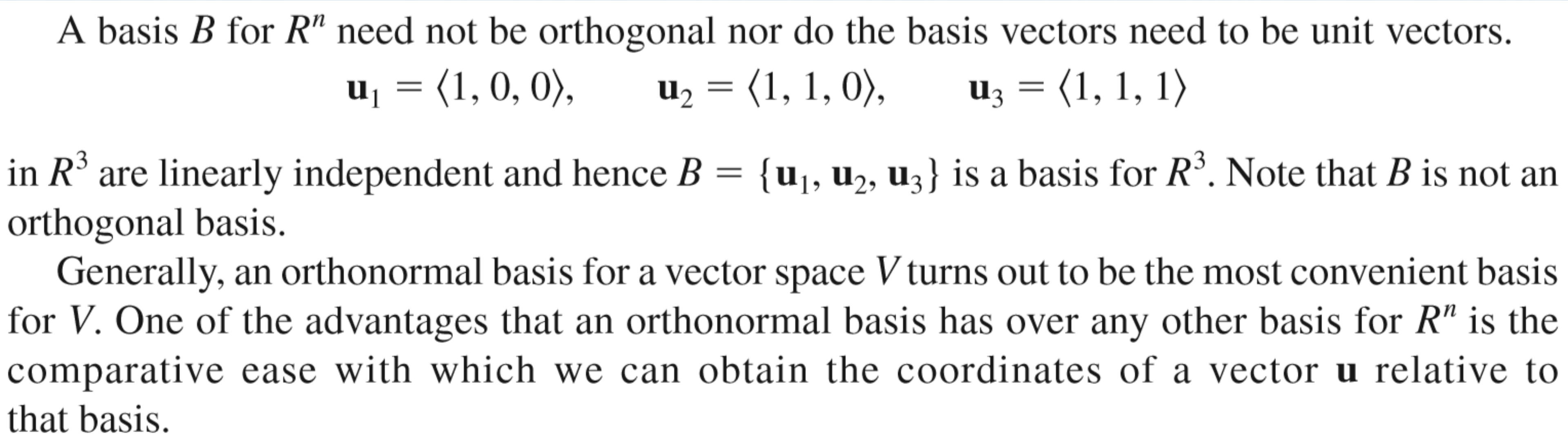
The set of three vectors

$$\mathbf{w}_1 = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle, \mathbf{w}_2 = \left\langle -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle, \mathbf{w}_3 = \left\langle 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle \quad (1)$$

is linearly independent and spans the space R^3. Hence $B = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is a basis for R^3. Using the standard inner product or dot product defined on R^3, observe

$$\mathbf{w}_1 \cdot \mathbf{w}_2 = 0, \mathbf{w}_1 \cdot \mathbf{w}_3 = 0, \mathbf{w}_2 \cdot \mathbf{w}_3 = 0, \quad \text{and} \quad \|\mathbf{w}_1\| = 1, \|\mathbf{w}_2\| = 1, \|\mathbf{w}_3\| = 1.$$

Hence B is an orthonormal basis. ≡



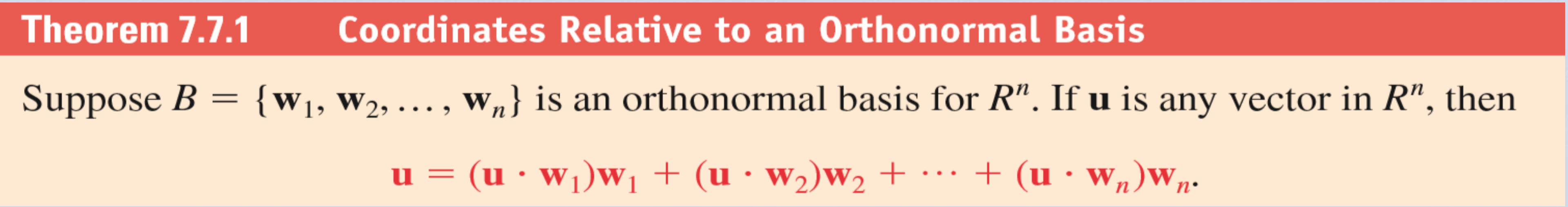
A basis B for R^n need not be orthogonal nor do the basis vectors need to be unit vectors.

$$\mathbf{u}_1 = \langle 1, 0, 0 \rangle, \qquad \mathbf{u}_2 = \langle 1, 1, 0 \rangle, \qquad \mathbf{u}_3 = \langle 1, 1, 1 \rangle$$

in R^3 are linearly independent and hence $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis for R^3. Note that B is not an orthogonal basis.

Generally, an orthonormal basis for a vector space V turns out to be the most convenient basis for V. One of the advantages that an orthonormal basis has over any other basis for R^n is the comparative ease with which we can obtain the coordinates of a vector $\mathbf{u}$ relative to that basis.

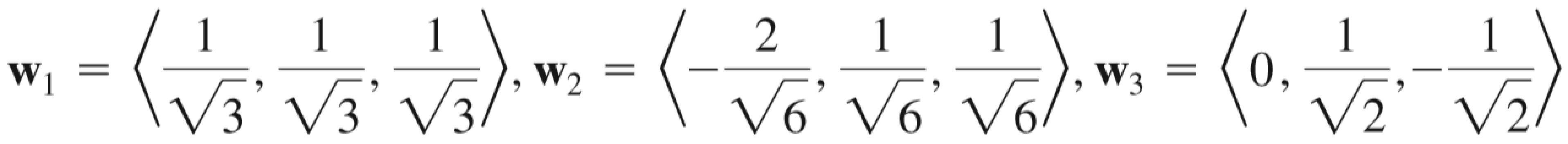
Orthonormal Basis



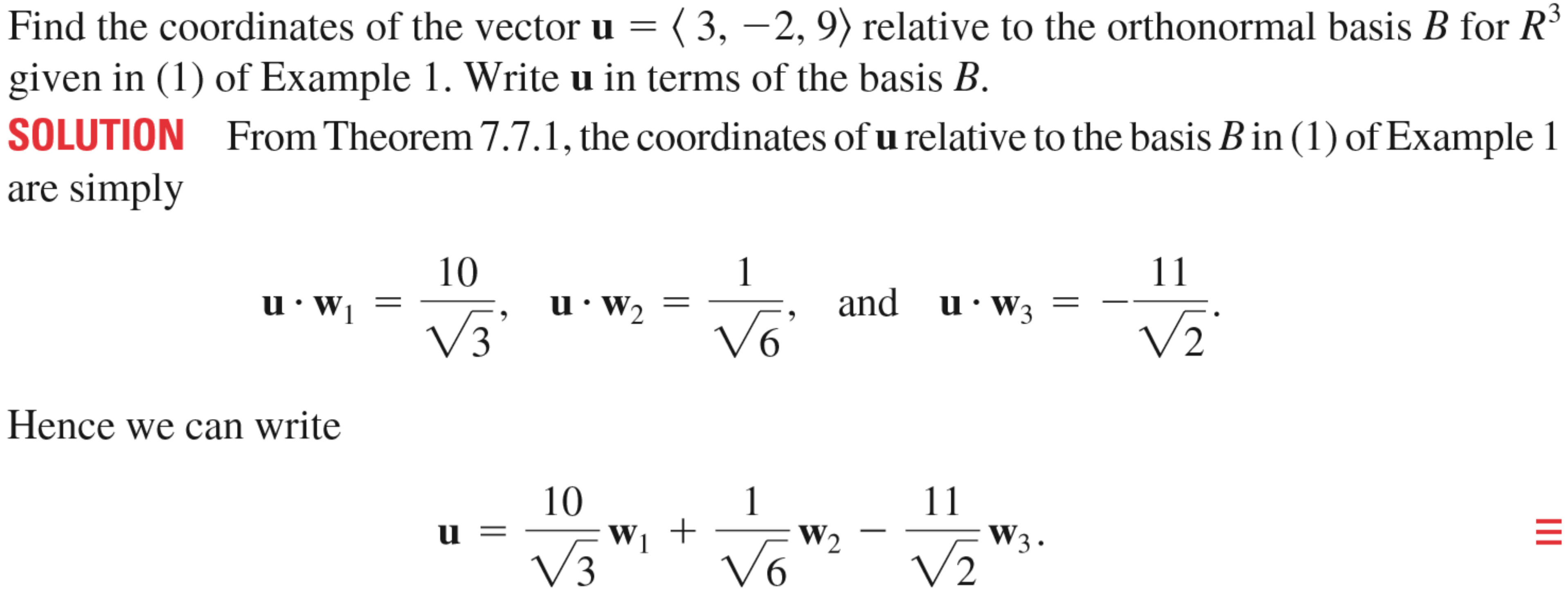
Theorem 7.7.1 Coordinates Relative to an Orthonormal Basis

Suppose $B = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ is an orthonormal basis for R^n. If $\mathbf{u}$ is any vector in R^n, then

$$\mathbf{u} = (\mathbf{u} \cdot \mathbf{w}_1)\mathbf{w}_1 + (\mathbf{u} \cdot \mathbf{w}_2)\mathbf{w}_2 + \cdots + (\mathbf{u} \cdot \mathbf{w}_n)\mathbf{w}_n.$$



$$\mathbf{w}_1 = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle, \mathbf{w}_2 = \left\langle -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle, \mathbf{w}_3 = \left\langle 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$$



Find the coordinates of the vector $\mathbf{u} = \langle 3, -2, 9 \rangle$ relative to the orthonormal basis B for R^3 given in (1) of Example 1. Write $\mathbf{u}$ in terms of the basis B.

SOLUTION From Theorem 7.7.1, the coordinates of $\mathbf{u}$ relative to the basis B in (1) of Example 1 are simply

$$\mathbf{u} \cdot \mathbf{w}_1 = \frac{10}{\sqrt{3}}, \quad \mathbf{u} \cdot \mathbf{w}_2 = \frac{1}{\sqrt{6}}, \quad \text{and} \quad \mathbf{u} \cdot \mathbf{w}_3 = -\frac{11}{\sqrt{2}}.$$

Hence we can write

$$\mathbf{u} = \frac{10}{\sqrt{3}}\mathbf{w}_1 + \frac{1}{\sqrt{6}}\mathbf{w}_2 - \frac{11}{\sqrt{2}}\mathbf{w}_3.$$

Gram–Schmidt Orthogonalization Process (แกรม-ชมิดท์ ออโธโกนอล โพรเซส)

- Gram– Schmidt orthogonalization process is an algorithm for generating an orthogonal basis $B' = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_\mathrm{n}\}$, from any given basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_\mathrm{n}\}$, for R^n

- How?

- Key idea in the orthogonalization process is vector projection
- (ทบทวน $\mathrm{proj}_\mathrm{b}\mathrm{a}$)
- Creating an orthonormal basis $B'' = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_\mathrm{n}\}$ by normalizing the vectors in the orthogonal basis B'

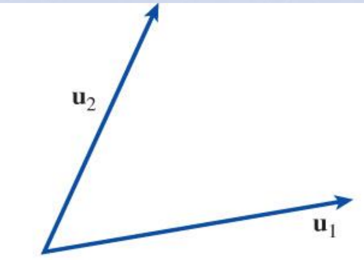
Gram–Schmidt Orthogonalization Process (Constructing an Orthogonal Basis for R^2)

- Transformation of a basis $B = \{\mathbf{u_1}, \mathbf{u_2}\}$, for R^2 into an orthogonal basis $B' = \{\mathbf{v_1}, \mathbf{v_2}\}$ consists of 2 steps.

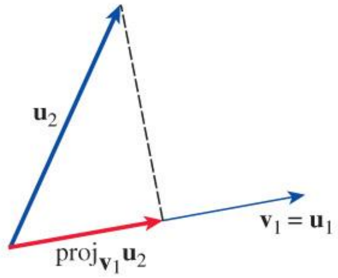
- The first step, we choose one of the vectors in B, say, $\mathbf{u_1}$, and rename it $\mathbf{v_1}$

- Next, we project the remaining vector $\mathbf{u_2}$ in B onto the vector $\mathbf{v_1}$ and define a second vector to be $\mathbf{v_2} = \mathbf{u_2} - \mathrm{proj}_{\mathbf{v_1}}\mathbf{u_2}$

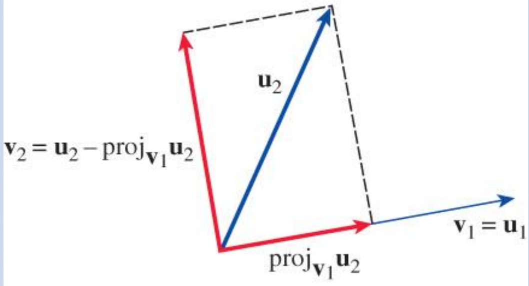
- $\mathrm{proj}_{\mathbf{v_1}}\mathbf{u_2} = \left(\frac{\mathbf{u_2}\cdot\mathbf{v_1}}{\mathbf{v_1}\cdot\mathbf{v_1}}\right)\mathbf{v_1}$



(a) Linearly independent vectors $\mathbf{u}_1$ and $\mathbf{u}_2$



(b) Projection of $\mathbf{u}_2$ onto $\mathbf{v}_1$



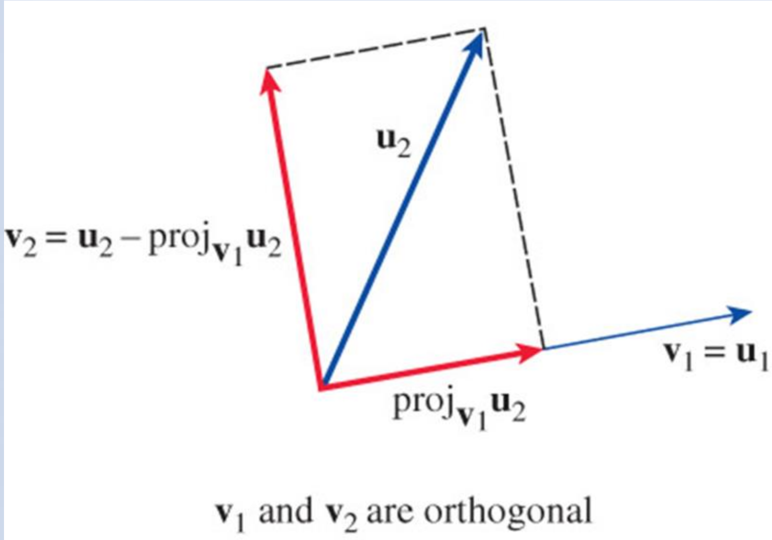
(c) $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal

Gram–Schmidt Orthogonalization Process (Constructing an Orthogonal Basis for R^2)

$$\mathbf{v}_1 = \mathbf{u}_1$$

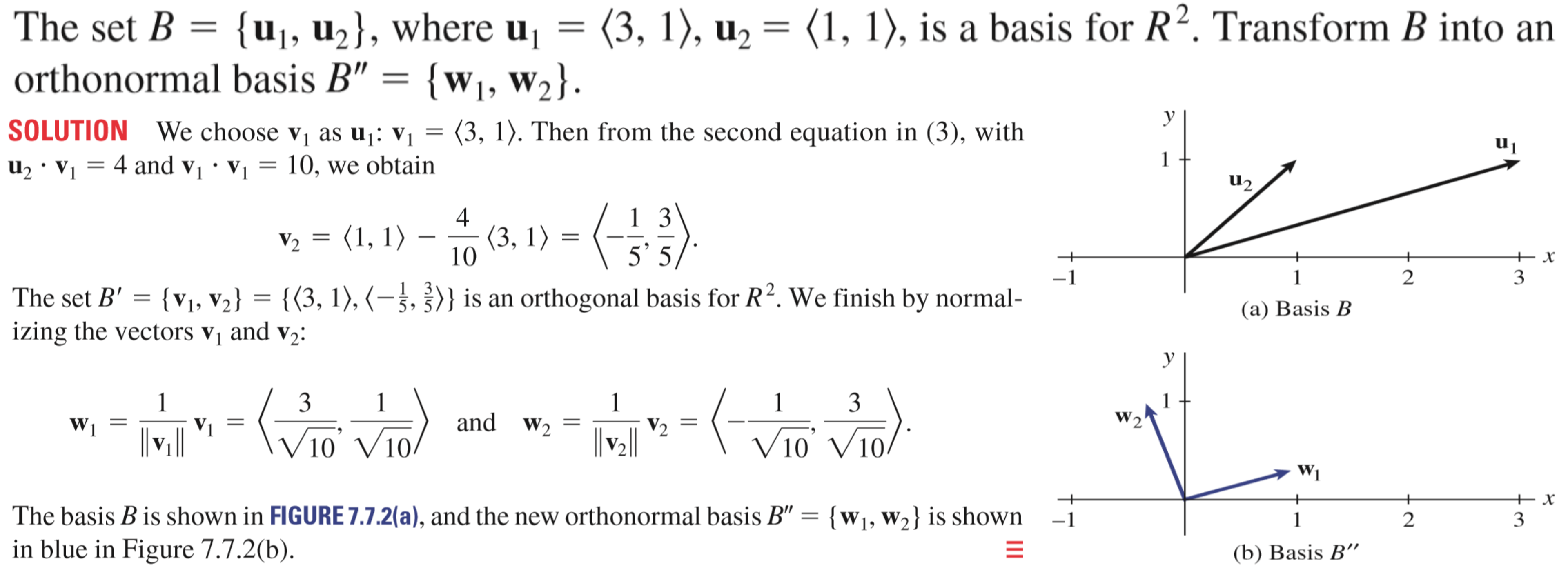
$$\mathbf{v}_2 = \mathbf{u}_2 - \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right) \mathbf{v}_1$$

$$\mathbf{v}_2 \cdot \mathbf{v}_1 = 0$$



$\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal

Gram–Schmidt Orthogonalization Process (Constructing an Orthogonal Basis for R^2)



The set $B = \{\mathbf{u}_1, \mathbf{u}_2\}$, where $\mathbf{u}_1 = \langle 3, 1\rangle$, $\mathbf{u}_2 = \langle 1, 1\rangle$, is a basis for R^2. Transform B into an orthonormal basis $B'' = \{\mathbf{w}_1, \mathbf{w}_2\}$.

SOLUTION We choose $\mathbf{v}_1$ as $\mathbf{u}_1$: $\mathbf{v}_1 = \langle 3, 1\rangle$. Then from the second equation in (3), with $\mathbf{u}_2 \cdot \mathbf{v}_1 = 4$ and $\mathbf{v}_1 \cdot \mathbf{v}_1 = 10$, we obtain

$$\mathbf{v}_2 = \langle 1, 1\rangle - \frac{4}{10}\langle 3, 1\rangle = \left\langle -\frac{1}{5}, \frac{3}{5}\right\rangle.$$

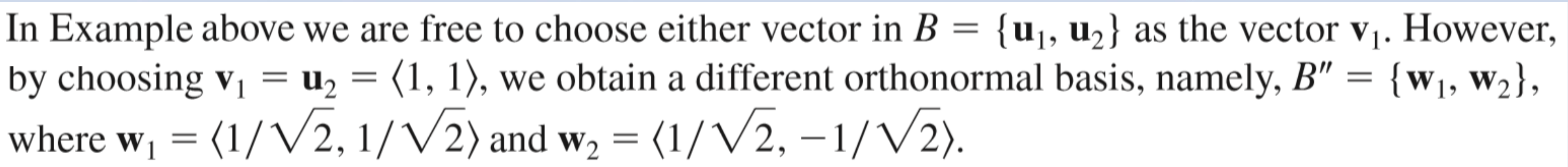
The set $B' = \{\mathbf{v}_1, \mathbf{v}_2\} = \{\langle 3, 1\rangle, \langle -\frac{1}{5}, \frac{3}{5}\rangle\}$ is an orthogonal basis for R^2. We finish by normalizing the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathbf{w}_1 = \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right\rangle \quad \text{and} \quad \mathbf{w}_2 = \frac{1}{\|\mathbf{v}_2\|}\mathbf{v}_2 = \left\langle -\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right\rangle.$$

The basis B is shown in **FIGURE 7.7.2(a)**, and the new orthonormal basis $B'' = \{\mathbf{w}_1, \mathbf{w}_2\}$ is shown in blue in Figure 7.7.2(b). ≡

(a) Basis B

(b) Basis B''



In Example above we are free to choose either vector in $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ as the vector $\mathbf{v}_1$. However, by choosing $\mathbf{v}_1 = \mathbf{u}_2 = \langle 1, 1\rangle$, we obtain a different orthonormal basis, namely, $B'' = \{\mathbf{w}_1, \mathbf{w}_2\}$, where $\mathbf{w}_1 = \langle 1/\sqrt{2}, 1/\sqrt{2}\rangle$ and $\mathbf{w}_2 = \langle 1/\sqrt{2}, -1/\sqrt{2}\rangle$.

A = [3 1;1 1]'; B = grams(A)

Gram–Schmidt Orthogonalization Process (Constructing an Orthogonal Basis for R^3)

Now suppose $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis for R^3.

Then the set $B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{v}_1 = \mathbf{u}_1$$

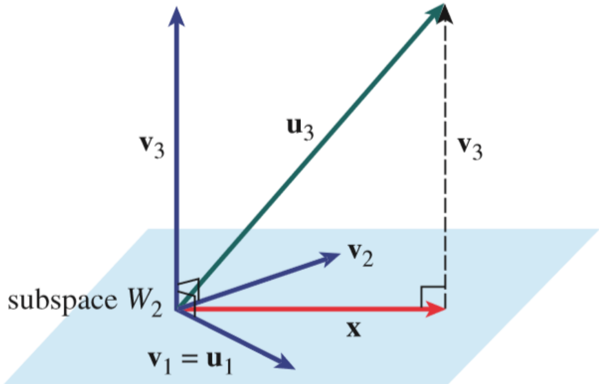
$\text{proj}_{\mathrm{v}_1}\mathrm{u}_2$

$$\mathbf{v}_2 = \mathbf{u}_2 - \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1$$

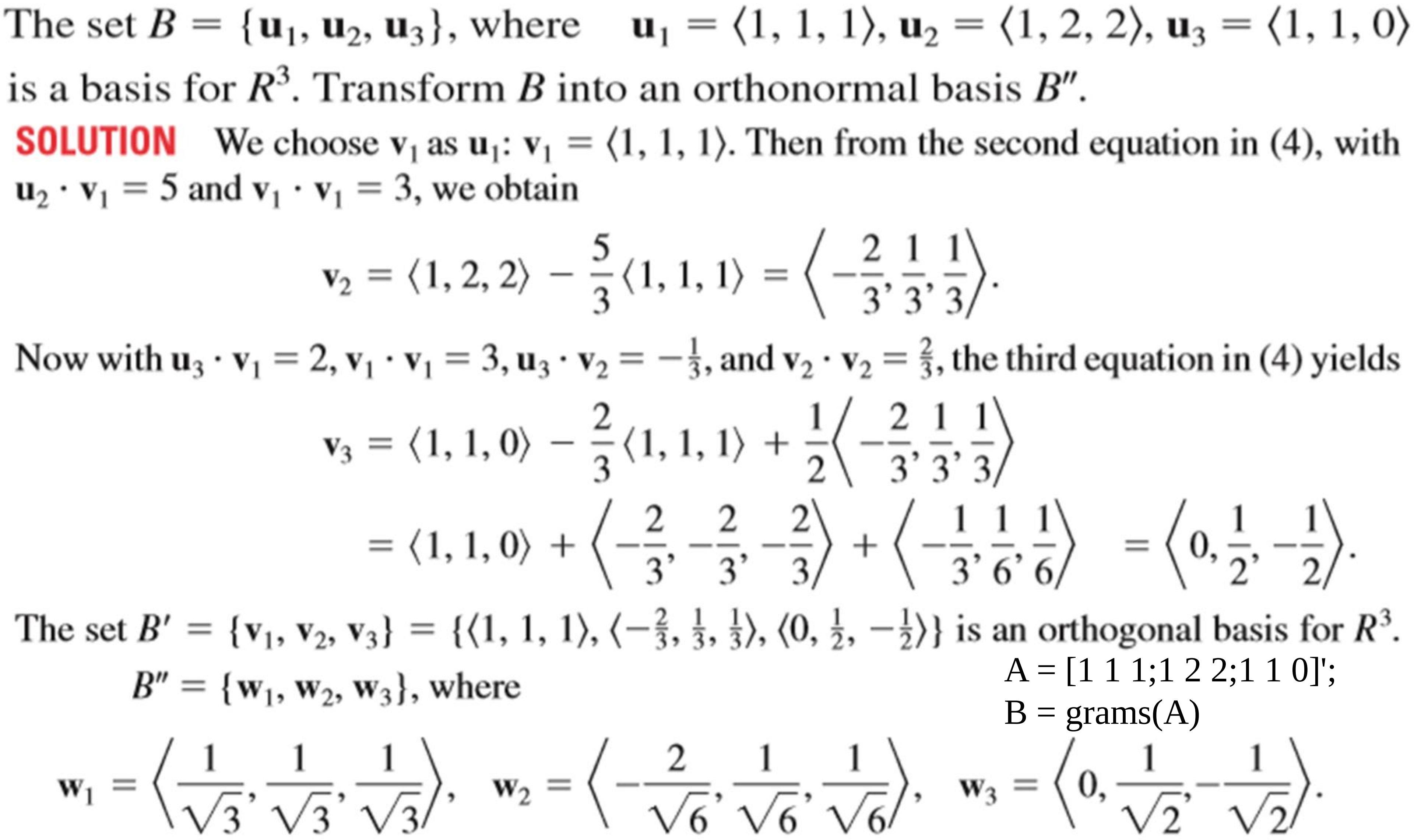
$\text{proj}_{\mathrm{v}_1}\mathrm{u}_3$

$$\mathbf{v}_3 = \mathbf{u}_3 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2$$

$\text{proj}_{\mathrm{v}_2}\mathrm{u}_3$



Gram–Schmidt Orthogonalization Process (Constructing an Orthogonal Basis for R^3)



The set $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where $\mathbf{u}_1 = \langle 1, 1, 1\rangle, \mathbf{u}_2 = \langle 1, 2, 2\rangle, \mathbf{u}_3 = \langle 1, 1, 0\rangle$ is a basis for R^3. Transform B into an orthonormal basis B''.

SOLUTION We choose $\mathbf{v}_1$ as $\mathbf{u}_1$: $\mathbf{v}_1 = \langle 1, 1, 1\rangle$. Then from the second equation in (4), with $\mathbf{u}_2 \cdot \mathbf{v}_1 = 5$ and $\mathbf{v}_1 \cdot \mathbf{v}_1 = 3$, we obtain

$$\mathbf{v}_2 = \langle 1, 2, 2\rangle - \frac{5}{3}\langle 1, 1, 1\rangle = \left\langle -\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right\rangle.$$

Now with $\mathbf{u}_3 \cdot \mathbf{v}_1 = 2, \mathbf{v}_1 \cdot \mathbf{v}_1 = 3, \mathbf{u}_3 \cdot \mathbf{v}_2 = -\frac{1}{3}$, and $\mathbf{v}_2 \cdot \mathbf{v}_2 = \frac{2}{3}$, the third equation in (4) yields

$$\begin{aligned}\mathbf{v}_3 &= \langle 1, 1, 0\rangle - \frac{2}{3}\langle 1, 1, 1\rangle + \frac{1}{2}\left\langle -\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right\rangle \\ &= \langle 1, 1, 0\rangle + \left\langle -\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}\right\rangle + \left\langle -\frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right\rangle \quad = \left\langle 0, \frac{1}{2}, -\frac{1}{2}\right\rangle.\end{aligned}$$

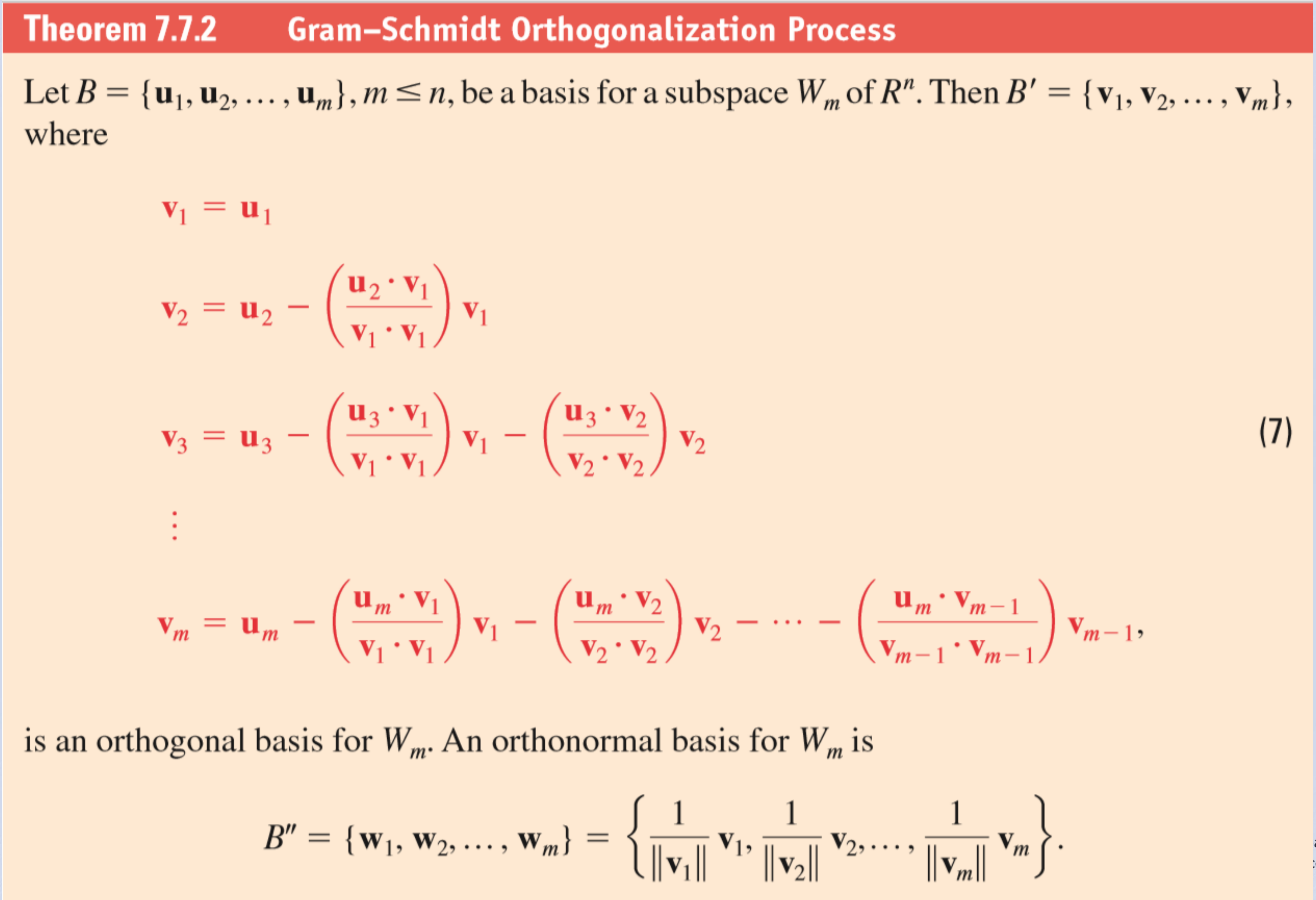
The set $B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{\langle 1, 1, 1\rangle, \langle -\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\rangle, \langle 0, \frac{1}{2}, -\frac{1}{2}\rangle\}$ is an orthogonal basis for R^3.

$B'' = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, where

$$\mathbf{w}_1 = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\rangle, \quad \mathbf{w}_2 = \left\langle -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right\rangle, \quad \mathbf{w}_3 = \left\langle 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\rangle.$$

```
A = [1 1 1;1 2 2;1 1 0]';
B = grams(A)
```

Gram–Schmidt Orthogonalization Process



Theorem 7.7.2 Gram–Schmidt Orthogonalization Process

Let $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$, $m \leq n$, be a basis for a subspace W_m of R^n. Then $B' = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$, where

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{u}_1 \\
\mathbf{v}_2 &= \mathbf{u}_2 - \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{u}_3 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 \\
&\vdots \\
\mathbf{v}_m &= \mathbf{u}_m - \left(\frac{\mathbf{u}_m \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{u}_m \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 - \cdots - \left(\frac{\mathbf{u}_m \cdot \mathbf{v}_{m-1}}{\mathbf{v}_{m-1} \cdot \mathbf{v}_{m-1}}\right)\mathbf{v}_{m-1},
\end{aligned} \tag{7}$$

is an orthogonal basis for W_m. An orthonormal basis for W_m is

$$B'' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\} = \left\{\frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1, \frac{1}{\|\mathbf{v}_2\|}\mathbf{v}_2, \ldots, \frac{1}{\|\mathbf{v}_m\|}\mathbf{v}_m\right\}.$$

Thanks

Gram–Schmidt Orthogonalization Process

- สมมุติว่า $B = \langle \mathbf{w_1}, \mathbf{w_2}, \dots, \mathbf{w_n} \rangle$เป็นฐานตั้งฉาก (Orthogonal basis) สำหรับ $\mathbf{R}^n$ และถ้า $\mathbf{u}$ เป็นเว็คเตอร์ใดๆใน $\mathbf{R}^n$, ดังนั้น

$$\mathbf{u} = (\mathbf{u} \cdot \mathbf{w_1})\mathbf{w_1} + (\mathbf{u} \cdot \mathbf{w_2})\mathbf{w_2} + \cdots + (\mathbf{u} \cdot \mathbf{w}_{\boldsymbol{n}})\mathbf{w}_{\boldsymbol{n}}$$

- ฐาน B ของ $\mathbf{R}^n$ สามารถถูกแปลงเป็นฐานที่ตั้งฉาก

$B' = \langle \mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v}_{\boldsymbol{n}} \rangle$ แล้วแปลงเป็นฐานที่ตั้งฉาก $B'' = \langle \mathbf{w_1}, \mathbf{w_2}, \dots, \mathbf{w}_{\boldsymbol{n}} \rangle$ โดยการทำนอร์มอลไลซ์เว็คเตอร์ใน B'

- เว็คเตอร์ $\mathbf{v}_n$ และ $\mathbf{w}_n$ เป็นเว็คเตอร์ที่ตั้งฉากกันและเป็นเว็คเตอร์หน่วย

Gram–Schmidt Orthogonalization Process

- <u>ตัวอย่าง:</u> การแปลงเซ็ทท $B = \{\mathbf{u_1}, \mathbf{u_2}\}$ ไปเป็นฐานตั้งฉาก $B'' = \{\mathbf{w_1}, \mathbf{w_2}\}$ (โดยที่ $\mathbf{u_1} = \langle 3,1\rangle$ และ $\mathbf{u_2} = \langle 1,1\rangle$)
  - เลือก $\mathbf{v_1} = \mathbf{u_1} = \langle 3,1\rangle$ และ $\mathbf{v_2} = \mathbf{u_2} - \left(\frac{\mathbf{u_2}\cdot\mathbf{v_1}}{\mathbf{v_1}\cdot\mathbf{v_1}}\right)\mathbf{v_1} = \langle 1,1\rangle - \frac{4}{10}\langle 3,1\rangle = \left\langle -\frac{1}{5}, \frac{3}{5}\right\rangle$
  - เซ็ทท $B' = \left\{\langle 3,1\rangle, \left\langle -\frac{1}{5}, \frac{3}{5}\right\rangle\right\}$ จะมีฐานตั้งฉากสำหรับ R^2

Gram–Schmidt Orthogonalization Process

- <u>ตัวอย่าง:</u>
  - ทำได้โดยการนอร์มัลไลซ์เว็คเตอร์ $\mathbf{v_1}$ และ $\mathbf{v_2}$

$$\mathbf{w_1} = \frac{\mathbf{v_1}}{\|\mathbf{v_1}\|} = \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle \text{ และ } \mathbf{w_2} = \frac{\mathbf{v_2}}{\|\mathbf{v_2}\|} = \left\langle -\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle$$

  - ฐานใหม่ที่ตั้งฉาก(new orthonormal basis) เท่ากับ
  - $B'' = \{\mathbf{w_1}, \mathbf{w_2}\} = \left\{ \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle, \left\langle -\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle \right\}$

ปริภูมิเวกเตอร์ (Vector Spaces)

- เซ็ทของเว็คเตอร์ $B = \langle \mathbf{x_1}, \mathbf{x_2}, \dots, \mathbf{x_n} \rangle$ ในปริภูมิเวกเตอร์ V คือเกณฑ์หลักสำหรับ V ถ้า
  - B เป็นอิสระแบบเชิงเส้น (linearly independent)
  - แต่ละเว็คเตอร์ใน V จะอยู่รูปผลรวมเชิงเส้นของเว็คเตอร์เหล่านั้น
  - เซ็ทของเว็คเตอร์ $\langle \mathbf{x_1}, \mathbf{x_2}, \dots, \mathbf{x_n} \rangle$ เป็นอิสระเชิงเส้น ถ้ามีค่าคงที่ที่สอดคล้องกับสมการด้านล่าง
  - $k_1\mathbf{x_1} + k_2\mathbf{x_2} + \cdots + k_n\mathbf{x_n} = \mathbf{0}$ เพราะ $k_1 = k_2 = \cdots k_n = 0$

1. Introduction to multi-variable calculus.
2. Polar coordinates.
3. Analysis of functions of several variables,
4. vector valued functions,
5. partial derivatives, and
6. multiple integrals.
7. Vector analysis.
8. Optimization techniques,
9. parametric equations,
10. line integrals,
11. surface integrals and
12. major theorems concerning their applications: Green’s theorem, Divergence theorem, Gauss theorem.
13. Complex variable.
14. Functions of a complex variable.
15. Derivatives and Cauchy- Riemann equations.
16. Integrals and Cauchy integral theorem.
17. Power and Laurent Series.
18. Residue theory.
19. Conformal mapping and
20. Fourier series applications.