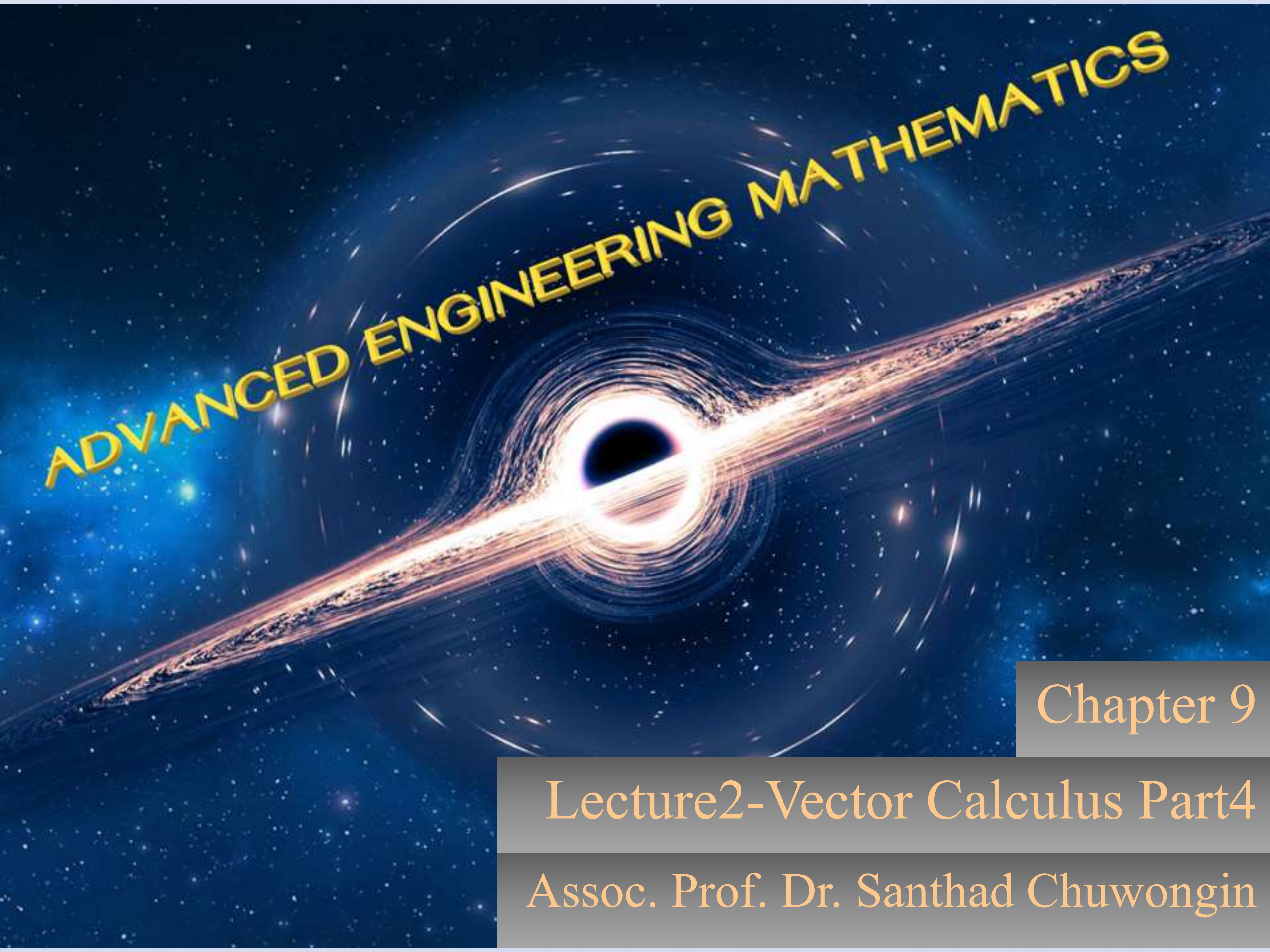


Chapter 9

Lecture2-Vector Calculus Part4

Assoc. Prof. Dr. Santhad Chuwongin

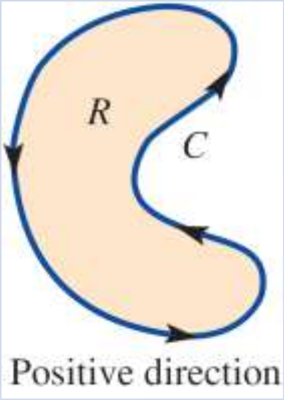
Outline

Green's Theorem (ทฤษฎีบทของกรีน)

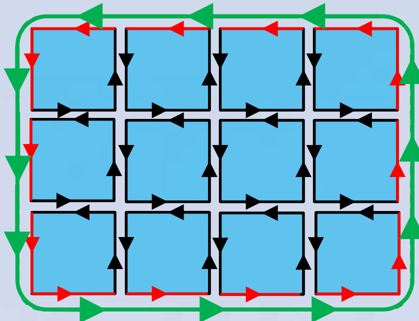
- ทฤษฎีบทของกรีน เป็นการแปลงอินทิกรัลตามเส้นโค้งเรียบรอบวงปิด C ไปเป็นอินทิกรัลสองชั้นเหนือบริเวณ R ที่ถูกปิดล้อมด้วย C (<u>**ข้อจำกัด**</u> ใช้ได้เฉพาะสนามเว็คเตอร์ ๒ มิติเท่านั้น)
- การหมุนวนบนระนาบ xy (หรือ $\mathbf{F} \cdot d\mathbf{r}$) คือ การหมุนรอบแกน z (หรือ $\nabla \times \mathbf{F}$)

Theorem 9.12.1 Green's Theorem in the Plane

กำหนดให้ C เป็นเส้นโค้งเรียบรอบวงปิดที่ต่อเนื่องในขอบเขต R ถ้า P, Q, $\dfrac{\partial P}{\partial y}$ และ $\dfrac{\partial Q}{\partial x}$ ต่อเนื่องในขอบเขต R โดยที่สนามเว็คเตอร์ $\mathbf{F}(x,y)=P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$, $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ ดังนั้น



$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy$$



เส้นโค้ง C

George Green (1793–1841)

George Green

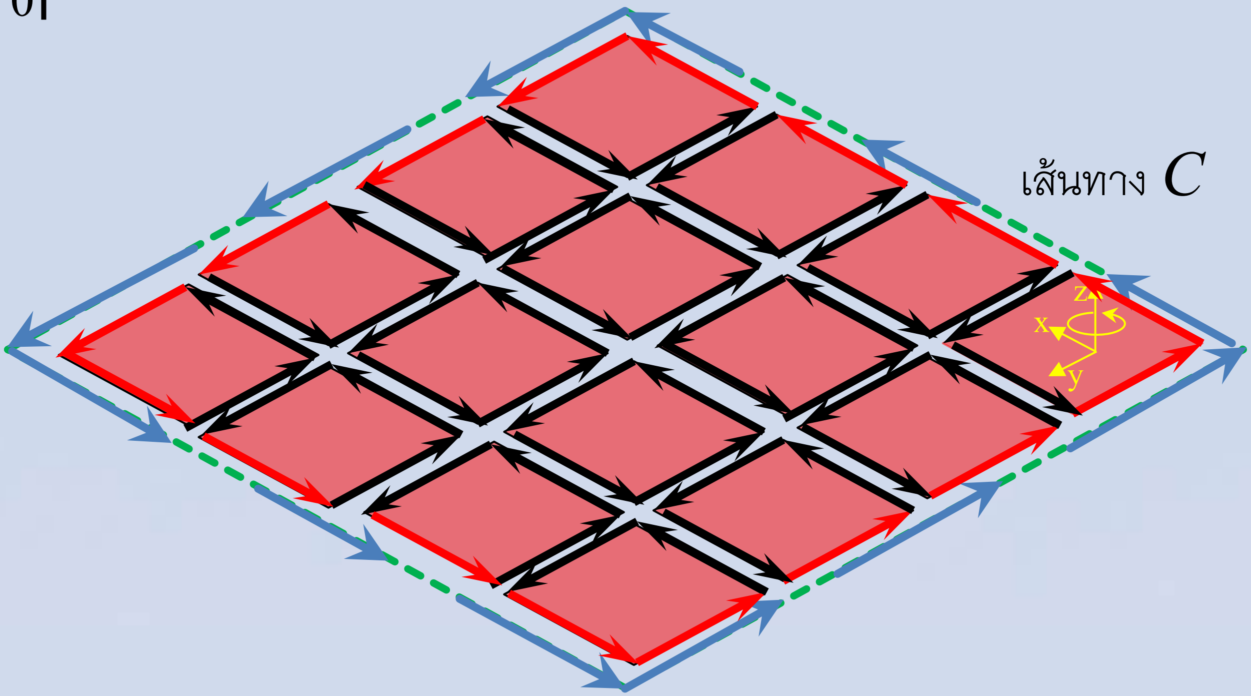
Green's Theorem is named after the self-taught English scientist George Green (1793–1841). He worked full-time in his father's bakery from the age of nine and taught himself mathematics from library books. In 1828 he published privately *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*, but only 100 copies were printed and most of those went to his friends. This pamphlet contained a theorem that is equivalent to what we know as Green's Theorem, but it didn't become widely known at that time.

Finally, at age 40, Green entered Cambridge University as an undergraduate but died four years after graduation. In 1846 William Thomson (Lord Kelvin) located a copy of Green's essay, realized its significan e, and had it reprinted. Green was the first person to try to formulate a mathematical theory of electricity and magnetism. His work was the basis for the subsequent electromagnetic theories of Thomson, Stokes, Rayleigh, and Maxwell.

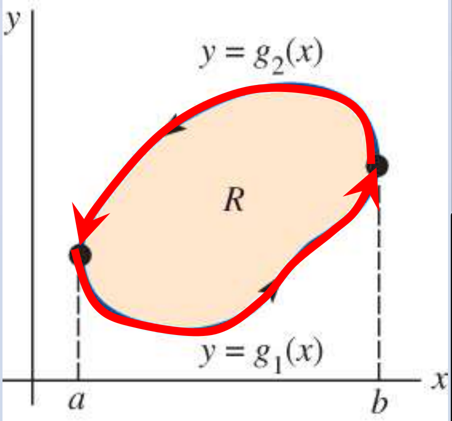
Green’s Theorem (ทฤษฎีบทของกรีน)

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy$$

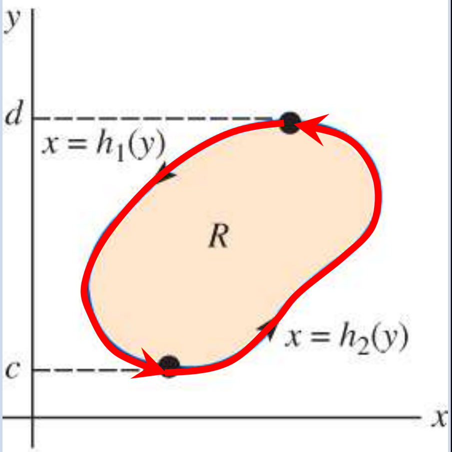
$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ P & Q & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$



Proof :Green's Theorem (พิสูจน์ทฤษฎีบทของกรีน)



(a) R as a Type I region



(b) R as a Type II region

$$\oint_C \boldsymbol{Pdx} + \boldsymbol{Qdy} = \iint_R \left(\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{x}} - \frac{\partial \boldsymbol{P}}{\partial \boldsymbol{y}}\right) \boldsymbol{dA}$$

$$-\iint_R \frac{\partial P}{\partial y} dA = -\int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y} dy dx = -\int_a^b \left[P\big(x, g_2(x)\big) - P\big(x, g_1(x)\big)\right] dx$$

$$= \int_a^b P\big(x, g_1(x)\big) dx + \int_b^a P\big(x, g_2(x)\big) dx = \oint_C P(x, y) dx$$

$$\iint_R \frac{\partial Q}{\partial x} dA = \int_c^d \int_{h_1(y)}^{h_2(y)} \frac{\partial Q}{\partial x} dx dy = \int_c^d [Q(h_2(y), y) - Q(h_1(y), y)] dy$$

$$= \int_c^d Q(h_2(y), y) dy + \int_d^c Q(h_1(y), y) dy = \oint_C Q(x, y) dy$$

$$\oint_C \boldsymbol{Pdx} + \boldsymbol{Qdy} = \iint_R \left(\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{x}} - \frac{\partial \boldsymbol{P}}{\partial \boldsymbol{y}}\right) \boldsymbol{dA}$$

Green's Theorem (ทฤษฎีบทของกรีน)

ตัวอย่าง จากทฤษฎีบทของกรีน

ถ้า C เป็นเส้นรอบวงกลม $(x-2)^2+y^2=1$

จงหาค่า $\oint_C \; ye^{-x}dx + \left(\frac{x^2}{2} - e^{-x}\right)dy$

```
import sympy as sp
t = sp.symbols('t')
x, y = 2 + sp.cos(t), sp.sin(t)
F = [y * sp.exp(-x), x**2 / 2 - sp.exp(-x)]
sol = sp.integrate(F[0] * sp.diff(x, t) + F[1] *
sp.diff(y, t), (t, 0, 2 * sp.pi))
print(sol)
```

วิธีทำ $P(x,y) = ye^{-x}, Q(x,y) = \frac{x^2}{2} - e^{-x}$ ดังนั้น $\frac{\partial P}{\partial y} = e^{-x}, \frac{\partial Q}{\partial x} = x + e^{-x}$

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_R x dA = \int_0^{2\pi}\int_0^1 (2 + r\cos\theta)\, r dr d\theta = 2\pi$$

วงกลม$(x-2)^2+y^2=1$ หรือ $x-2 = r\cos\theta \;, y = r\sin\theta\,, dx = -r\sin\theta\, d\theta, dy = r\cos\theta\, d\theta$

$$\oint_C (ye^{-x})dx + \left(\frac{x^2}{2} - e^{-x}\right)dy = \int_0^{2\pi} -\left(\sin\theta\, e^{-(2+\cos\theta)}\right)\sin\theta\, d\theta + \left(\frac{4 + 4\cos\theta + \cos^2\theta}{2} - e^{-(2+\cos\theta)}\right)\cos\theta\, d\theta = ?$$

```
import sympy as sp
x, y = sp.symbols('x y')
F = [y * sp.exp(-x), x**2 / 2 - sp.exp(-x)]
dQdx_dPdy = sp.diff(F[1], x) - sp.diff(F[0], y)
sol = sp.integrate(sp.integrate(dQdx_dPdy, (x, 2 - sp.sqrt(1-y**2), 2 + sp.sqrt(1-y**2))), (y, -1, 1))
print(sol)
```

HW
Green’s Theorem

In Problems 21 and 22, evaluate $\oint_C x^2y^3\,dx - xy^2\,dy$ on the given closed curve C.

21.

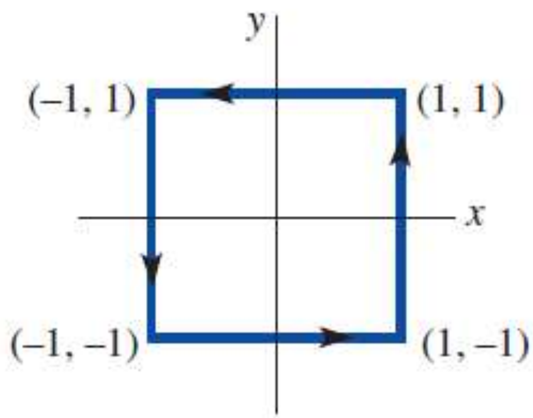


FIGURE 9.8.17 Closed curve C for Problem 21

22.

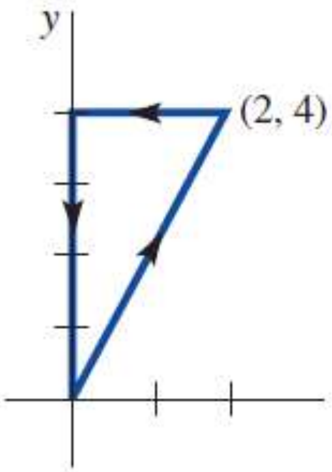


FIGURE 9.8.18 Closed curve C for Problem 22

```
import sympy as sp
x, y = sp.symbols('x y')
F = [x**2 * y**3, -x * y**2]
curl = sp.diff(F[1], x) - sp.diff(F[0], y) #dQdx-dPdy
sol = sp.integrate(sp.integrate(curl,(x,-1,1)),(y,-1,1))
print(sol) #21
```

```
import sympy as sp
x, y = sp.symbols('x y')
F = [x**2 * y**3, -x * y**2]
curl = sp.diff(F[1], x) - sp.diff(F[0], y) #dQdx-dPdy
sol = sp.integrate(sp.integrate(curl,(x,0,y/2)),(y,0,4))
print(sol) #22
```

Application : Finding Areas with Green's Theorem (การประยุกต์ใช้หาพื้นที่ด้วยทฤษฎีบทของกรีน)

Another application of the reverse direction of Green's Theorem is in computing areas. Since the area of D is $\iint_D 1\, dA$, we wish to choose P and Q so that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

There are several possibilities:

$$P(x, y) = 0 \qquad P(x, y) = -y \qquad P(x, y) = -\tfrac{1}{2}y$$

$$Q(x, y) = x \qquad Q(x, y) = 0 \qquad Q(x, y) = \tfrac{1}{2}x$$

Then Green's Theorem gives the following formulas for the area of D:

5 $$A = \oint_C x\, dy = -\oint_C y\, dx = \frac{1}{2}\oint_C x\, dy - y\, dx$$

Application : Finding Areas with Green's Theorem (การประยุกต์ใช้หาพื้นที่ด้วยทฤษฎีบทของกรีน)

ตัวอย่าง หาพื้นที่วงกลมรัศมี r หน่วยจากอินทิกรัลตามเส้น โดยใช้ทฤษฎีบทของกรีน

$$\oint_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \leftrightarrow \oint_C Pdx + Qdy = \iint_R 1\, dA$$

เพื่อหาพื้นที่วงกลมรัศมี r หน่วย, $\frac{\partial P}{\partial y} = 0$ และ $\frac{\partial Q}{\partial x} = 1$ ซึ่งจะทำให้ $P = 0, Q = x$ ดังนั้นจากทฤษฎีบทของกรีน จะได้ว่า

$$\oint_C xdy = \iint_R 1\, dA$$

จากนั้นใช้สมการพาราเมตริกของวงกลม 1 หนึ่งหน่วย คือ $x = r\cos\theta\,, y = r\sin\theta$ โดย $0 \le \theta \le 2\pi$ ซึ่งเป็นการเปลี่ยนตัวแปร x, y ให้อยู่ในรูป θ

$$Area = \oint_C xdy = r^2 \int_0^{2\pi} cos^2\theta\, d\theta = r^2 \int_0^{2\pi} \left(\frac{1+\cos 2\theta}{2}\right) d\theta = \frac{r^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_0^{2\pi} = \pi r^2$$

Application : Finding Areas with Green's Theorem
(การประยุกต์ใช้หาพื้นที่ด้วยทฤษฎีบทของกรีน)

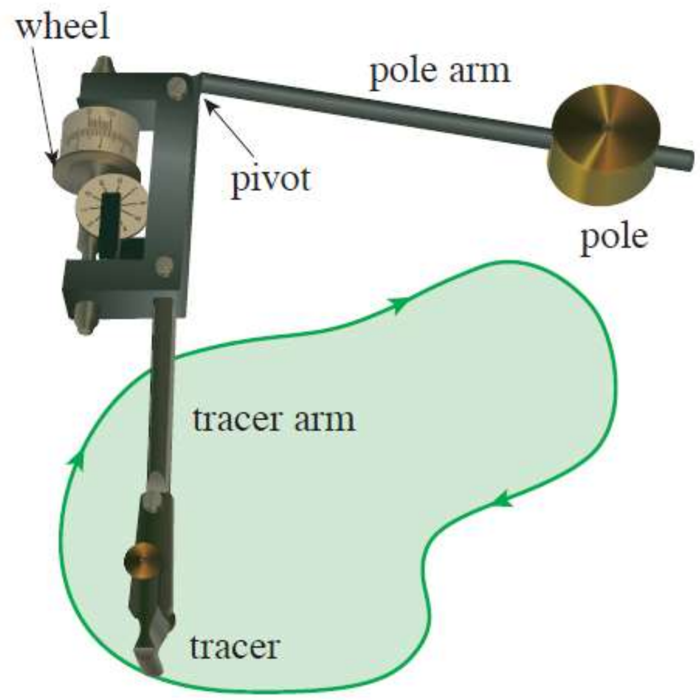
EXAMPLE Find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

SOLUTION The ellipse has parametric equations $x = a\cos t$ and $y = b\sin t$, where $0 \leqslant t \leqslant 2\pi$. Using the third formula in Equation 5, we have

$$\begin{aligned}
A &= \tfrac{1}{2}\int_C x\,dy - y\,dx \\
&= \tfrac{1}{2}\int_0^{2\pi} (a\cos t)(b\cos t)\,dt - (b\sin t)(-a\sin t)\,dt \\
&= \frac{ab}{2}\int_0^{2\pi} dt = \pi ab
\end{aligned}$$

A ***planimeter*** is an ingenious mechanical instrument invented in the 19th century for measuring the area of a region by tracing its boundary curve. For instance, a biologist could use one of these devices to measure the surface area of a leaf or bird wing.

Figure shows the operation of a polar planimeter: the pole is fixed and, as the tracer is moved along the boundary curve of the region, the wheel partly slides and partly rolls perpendicular to the tracer arm. The planimeter measures the distance that the wheel rolls and this is proportional to the area of the enclosed region.

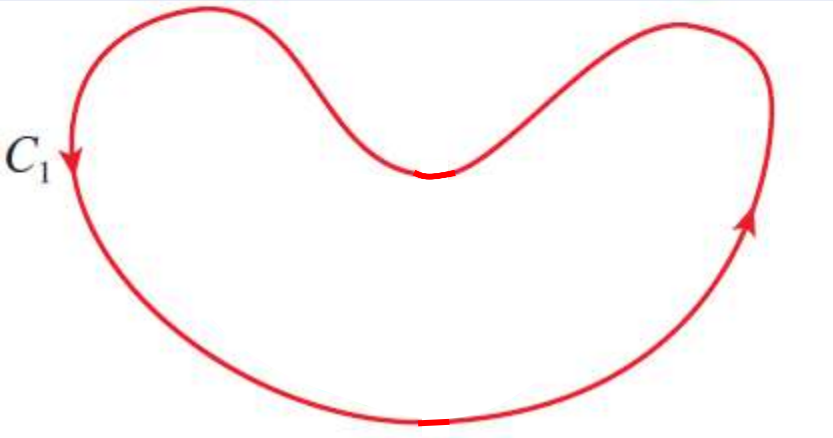


FIGURE

A Keuffel and Esser polar planimeter

- R. W. Gatterman, "The planimeter as an example of Green's Theorem" *Amer. Math. Monthly,* Vol. 88 (1981), pp. 701–4.
- Tanya Leise, "As the planimeter wheel turns" *College Math. Journal,* Vol. 38 (2007), pp. 24–31.

Extended Versions of Green's Theorem
(เพิ่มเติมทฤษฎีบทของกรีน)



$$\int_{C_1 \cup C_3} P\,dx + Q\,dy = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

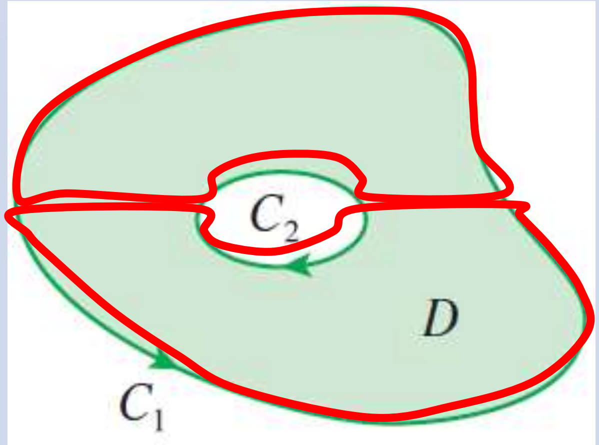
$$\int_{C_2 \cup (-C_3)} P\,dx + Q\,dy = \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

If we add these two equations, the line integrals along C_3 and $-C_3$ cancel, so we get

$$\int_{C_1 \cup C_2} P\,dx + Q\,dy = \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

which is Green's Theorem for $D = D_1 \cup D_2$, since its boundary is $C = C_1 \cup C_2$.

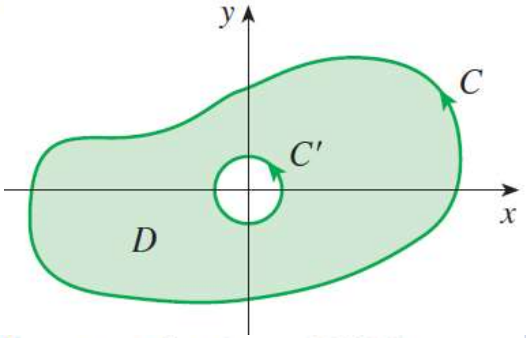
Extended Versions of Green's Theorem
(เพิ่มเติมทฤษฎีบทของกรีน)



$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{D'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA + \iint_{D''} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \int_{\partial D'} P\,dx + Q\,dy + \int_{\partial D''} P\,dx + Q\,dy$$

EXAMPLE 5 If $\mathbf{F}(x, y) = (-y\,\mathbf{i} + x\,\mathbf{j})/(x^2 + y^2)$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.



Since C is an *arbitrary* closed path that encloses the origin, it's difficult to compute the given integral directly.
So let's consider a counterclockwise-oriented circle C' with center the origin and radius a, where a is chosen to be small enough that C' lies inside C. (See Figure 11.) Let D be the region bounded by C and C'. Then its positively oriented boundary is $C \cup (-C')$ and so the general version of Green's Theorem gives

$$\int_C P\,dx + Q\,dy + \int_{-C'} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D \left[\frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{y^2 - x^2}{(x^2 + y^2)^2}\right] dA = 0$$

Therefore $\int_C P\,dx + Q\,dy = \int_{C'} P\,dx + Q\,dy$ that is, $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r}$

We now easily compute this last integral using the parametrization given by $\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j}$, $0 \leqslant t \leqslant 2\pi$. Thus

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_0^{2\pi} \frac{(-a\sin t)(-a\sin t) + (a\cos t)(a\cos t)}{a^2\cos^2 t + a^2\sin^2 t}\,dt = \int_0^{2\pi} dt = 2\pi$$

Green's Theorem (ทฤษฎีบทของกรีน)

ตัวอย่าง จากทฤษฎีบทของกรีน ถ้า C เป็นเส้นรอบวงกลม $x^2 + y^2 = 1$ จงพิสูจน์ว่า

$$\oint_C (y^2 - 7y)dx + (2xy + 2x)dy = \iint_R \left(\frac{\partial(2xy + 2x)}{\partial x} - \frac{\partial(y^2 - 7y)}{\partial y} \right) dA$$

วิธีทำ วงกลม $x^2 + y^2 = 1$ หรือ $x = \cos\theta$, $y = \sin\theta$, $dx = -\sin\theta\, d\theta$, $dy = \cos\theta\, d\theta$

$$\oint_C (y^2 - 7y)dx + (2xy + 2x)dy =$$

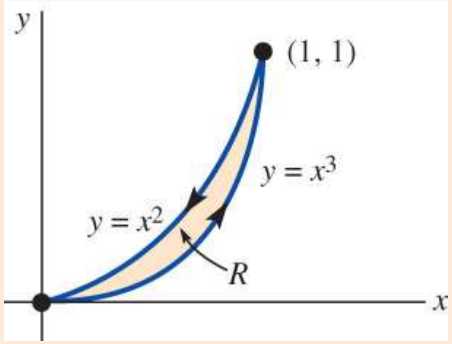
```
import sympy as sp
# Define symbols
x, y, t = sp.symbols('x y t', real=True)
# Define the vector field components
P = y**2 - 7*y
Q = 2*x*y + 2*x
# Parameterize the circle x² + y² = 1
x_param = sp.cos(t)
y_param = sp.sin(t)
# Substitute parameterization into P and Q
P_param = P.subs([(x, x_param), (y, y_param)])
Q_param = Q.subs([(x, x_param), (y, y_param)])
# Compute derivatives dx/dt and dy/dt
dx_dt = sp.diff(x_param, t)
dy_dt = sp.diff(y_param, t)
# Form the integrand: P(dx/dt) + Q(dy/dt)
integrand = P_param * dx_dt + Q_param * dy_dt
# Evaluate the integral from 0 to 2π
result = sp.integrate(integrand, (t, 0, 2*sp.pi))
```

Green’s Theorem (ทฤษฎีบทของกรีน)

```
import sympy as sp
# Define symbols
x, y, t = sp.symbols('x y t', real=True)

# Define the vector field components
P = y**2 - 7*y
Q = 2*x*y + 2*x

# Parameterize the circle x² + y² = 1
x_param = sp.cos(t)
y_param = sp.sin(t)

# Green's Theorem
dQ_dx = sp.diff(Q, x)
dP_dy = sp.diff(P, y)

# Convert to polar coordinates for the double integral
r, theta = sp.symbols('r theta', real=True, positive=True)

curl = dQ_dx - dP_dy
curl_polar = curl.subs([(x, r*sp.cos(theta)), (y, r*sp.sin(theta))])

# Double integral over the unit disk
integrand_polar = curl_polar * r
result_green = sp.integrate(integrand_polar, (r, 0, 1), (theta, 0, 2*sp.pi))
print(f"\nUsing Green's Theorem: {result_green}")
```

Green's Theorem (ทฤษฎีบทของกรีน)

ตัวอย่าง จงหาค่า $\oint_C (x^2 - y^2)dx + (2y - x)dy$ โดยที่ C อยู่ในควอดรันต์ที่ 1 เป็นเส้นทางที่ถูกปิดล้อมด้วยกราฟ $y = x^2$ และ $y = x^3$



จากโจทย์ $P(x, y) = x^2 - y^2$ และ $Q(x, y) = 2y - x$

ดังนั้น $\frac{\partial P}{\partial y} = -2y$ และ $\frac{\partial Q}{\partial x} = -1$

$$\oint_C (x^2 - y^2)dx + (2y - x)dy$$

```
import sympy as sp
# Define symbolic variables
x, y = sp.symbols('x y')

# Define the vector field
F = [x**2 - y**2, 2*y - x]

# Compute curl=(dQ/dx - dP/dy)
curl = sp.diff(F[1], x) - sp.diff(F[0], y)

# Double integral over the given region
C1 = sp.integrate(sp.integrate(curl, (y, x**3,
x**2)), (x, 0, 1))
print(C1)
```

Green's Theorem (ทฤษฎีบทของกรีน)

หรือ

$$\oint_C (x^2 - y^2)dx + (2y - x)dy = \int_{C1} + \int_{C2} =$$

$$= \int_0^1 (x^2 - x^6)dx + (2x^3 - x)3x^2dx + \int_1^0 (x^2 - x^4)dx + (2x^2 - x)2xdx$$

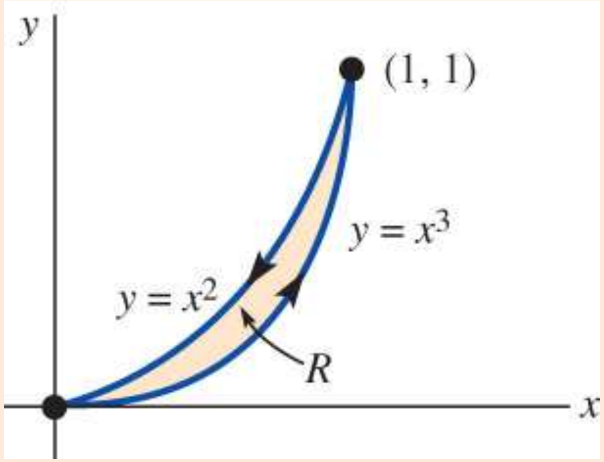
$$= \left(\frac{x^3}{3} - \frac{x^7}{7} + x^6 - \frac{3x^4}{4}\right)\Bigg|_0^1 + \left(\frac{x^3}{3} - \frac{x^5}{5} + x^4 - \frac{2x^3}{3}\right)\Bigg|_1^0$$

$$= \left(\frac{1}{3} - \frac{1}{7} + 1 - \frac{3}{4}\right) - \left(\frac{1}{3} - \frac{1}{5} + 1 - \frac{2}{3}\right)$$

$$= -\frac{1}{7} + \frac{1}{5} - \frac{3}{4} + \frac{2}{3}$$

$$= \frac{-60 + 84 - 315 + 280}{420}$$

$$= -\frac{11}{420}$$



```python
import sympy as sp

# Define symbolic variables
x, y = sp.symbols('x y')

# Define the vector field F
F = [x**2 - y**2, 2*y - x]

# Define parameterized curves
C1 = [x, x**3]  # First boundary curve
C2 = [x, x**2]  # Second boundary curve

# Compute derivatives of parameterized curves
dC1 = [sp.diff(C1[0], x), sp.diff(C1[1], x)]
dC2 = [sp.diff(C2[0], x), sp.diff(C2[1], x)]

# Compute line integrals along each curve
sol1 = sp.integrate(sp.Matrix(F).subs({x: C1[0], y:
C1[1]}).dot(sp.Matrix(dC1)), (x, 0, 1))
sol2 = sp.integrate(sp.Matrix(F).subs({x: C2[0], y:
C2[1]}).dot(sp.Matrix(dC2)), (x, 1, 0))

# Compute total integral
sol = sol1 + sol2

# Print the final result
print(sol)
```

Green's Theorem (ทฤษฎีบทของกรีน)

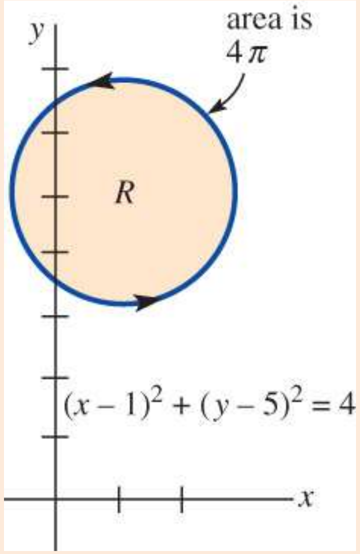
ตัวอย่าง จงหาค่า $\oint_C (x^5 + 3y)dx + \left(2x - e^{y^3}\right)dy$ โดยที่ C เป็นวงกลม$(x-1)^2+(y-5)^2=4$

ถ้า $P(x,y) = x^5 + 3y$และ $Q(x,y) = 2x - e^{y^3}$

ดังนั้น $\frac{\partial P}{\partial y} = 3$และ $\frac{\partial Q}{\partial x} = 2$

$$\oint_C (x^5 + 3y)dx + \left(2x - e^{y^3}\right)dy$$

เนื่องจากพื้นที่ของวงกลม คือ $\pi r^2 = 4\pi$




```
import sympy as sp
# Define symbols
x, y = sp.symbols('x y')
# Define the vector field F
F = [x**5 + 3*y, 2*x - sp.exp(y**3)]

# Compute dQ/dx - dP/dy
curl = sp.diff(F[1], x) - sp.diff(F[0], y)

# Define integration limits
y_lower = 5 - sp.sqrt(4 - (x - 1)**2)
y_upper = 5 + sp.sqrt(4 - (x - 1)**2)
# Compute the double integral
sol = sp.integrate(sp.integrate(curl,(y,y_lower,y_upper)),(x,-1,3))
print(sol)
```

Green's Theorem (ทฤษฎีบทของกรีน)

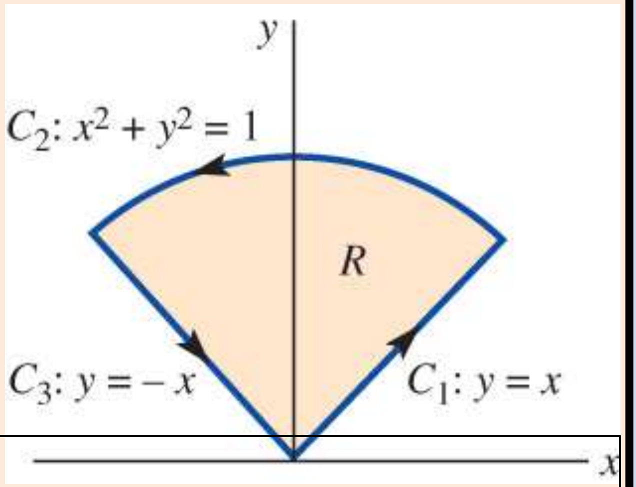
ตัวอย่าง จงหางานที่ถูกทำโดยแรง $\mathbf{F} = (-16y + \sin x^2)\mathbf{i} + (4e^y + 3x^2)\mathbf{j}$ ตามเส้นโค้งปิด C ดังรูป

$$W = \oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C (-16y + \sin x^2)dx + (4e^y + 3x^2)dy$$

และ ดังนั้นด้วยทฤษฎีบทของกรีน $\iint_R\ (6x + 16)dA$ จากรูปขอบเขต R ควรใช้ Polar coordinates เนื่องจาก R คือ

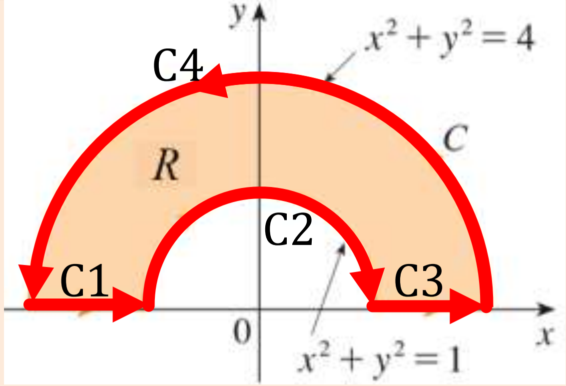
$$0 \le r \le 1, \frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$$

$$W = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^1 (6r\cos\theta + 16)\, r dr d\theta = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} [2r^3\cos\theta + 8r^2]_0^1 d\theta$$




```
import sympy as sp
x, y, r, t = sp.symbols('x y r t')
# Define the vector field
F = [-16*y + sp.sin(x**2), 4*sp.exp(y) + 3*x**2]
# Define the Contour parameterization
C = [r*sp.cos(t), r*sp.sin(t)]
# Substitute contour coordinates into dQdx_dPdy
curl = sp.diff(F[1], x) - sp.diff(F[0], y)
integrand = curl.subs({x: C[0], y: C[1]})* r
sol = (sp.integrate(sp.integrate(integrand, (r, 0, 1)), (t, sp.pi/4,
sp.pi*3/4)))
print(sol)
```

ตัวอย่าง จงหาค่า $\oint_C \; y^2dx + 3xy\,dy$ โดยที่ C เป็นขอบเขตของ R ดังรูป



วิธีทำ วงกลม $x^2 + y^2 = 1$ หรือ $x = cos\,\theta$ $, y = sin\,\theta\, , dx = -\,sin\,\theta\, d\theta, dy = cos\,\theta\, d\theta$

วงกลม $x^2 + y^2 = 4$ หรือ $x = 2\,cos\,\theta$ $, y = 2\,sin\,\theta\, , dx = -2\,sin\,\theta\, d\theta, dy = 2\,cos\,\theta\, d\theta$

$$\oint_C y^2dx + 3xy\,dy = \int_{C1} + \int_{C2} + \int_{C3} + \int_{C4}$$

```
import sympy as sp

# Define symbols
x, y, r, t = sp.symbols('x y r t')

# Define the vector field
F = sp.Matrix([y**2, 3*x*y])

# Compute dQ/dx - dP/dy
dQdx_dPdy = sp.diff(F[1], x) - sp.diff(F[0], y)

# Define the C contour parameterization
C = sp.Matrix([r*sp.cos(t), r*sp.sin(t)])

# Substitute contour coordinates into dQdx_dPdy
A = dQdx_dPdy.subs({x: C[0], y: C[1]})

# Compute the surface element dS
r_deriv = sp.diff(C, r)
t_deriv = sp.diff(C, t)
dS = r_deriv[0]*t_deriv[1] - r_deriv[1]*t_deriv[0]

# Compute the surface integral
result = sp.integrate(sp.integrate(A * dS, (r, 1, 2)), (t, 0, sp.pi))

print(result)
```