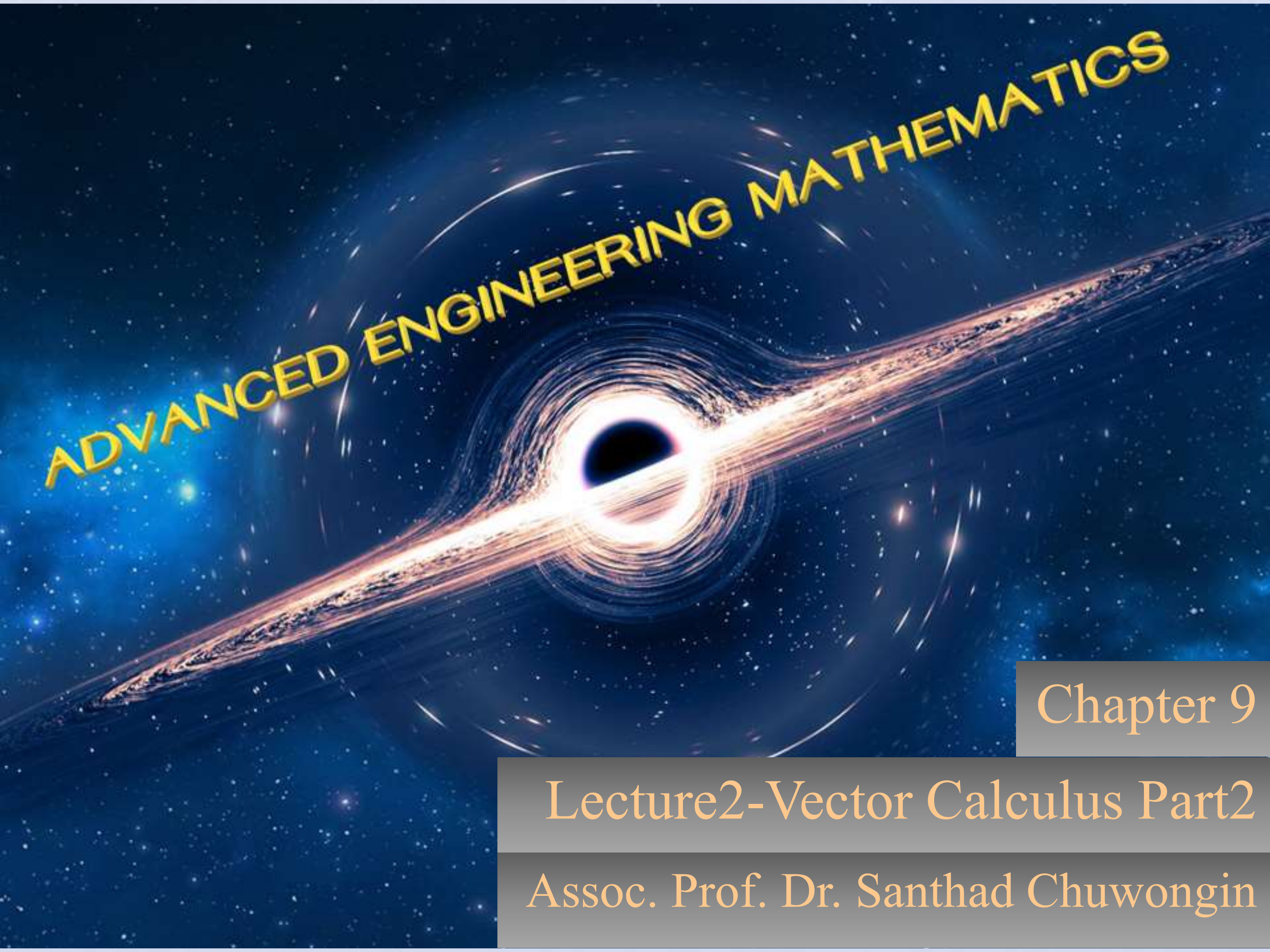
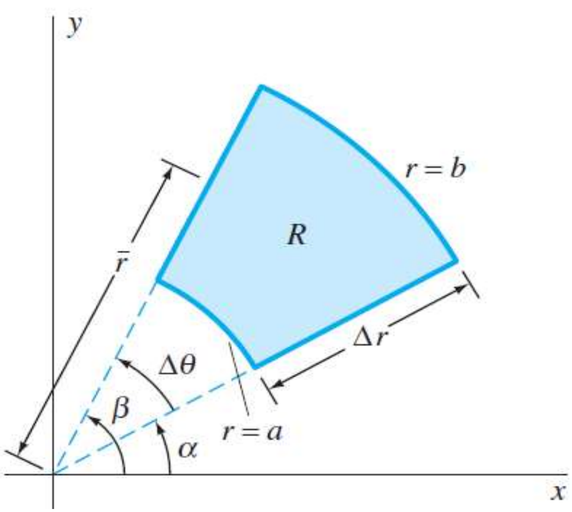

ADVANCED ENGINEERING MATHEMATICS
Chapter 9
Lecture2-Vector Calculus Part2
Assoc. Prof. Dr. Santhad Chuwongin

Outline

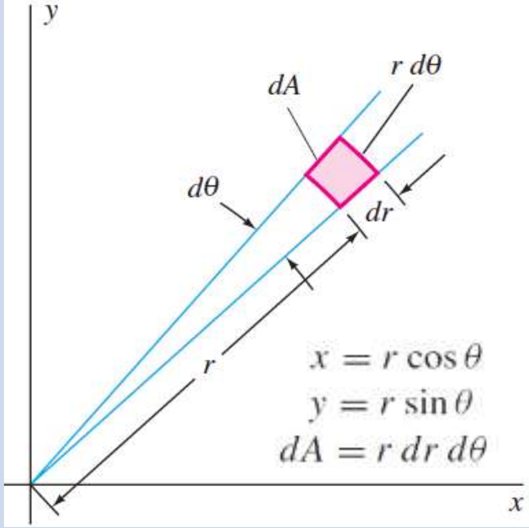
Double Integrals in Polar Coordinates
(อินทิกรัลสองชั้นในพิกัดเชิงขั้ว)

Double integral may be easier to evaluate (Transform rectangular *xy*-co. into polar *rθ*-co.)



$a \leqq r \leqq b, \qquad \alpha \leqq \theta \leqq \beta.$

Polar rectangle



Small polar rectangle area is

$dA = rdrd\theta$

Change of Variables: Rectangular to Polar Coordinates (การเปลี่ยนตัวแปรจากพิกัดฉากไปเป็นพิกัดเชิงขั้ว)

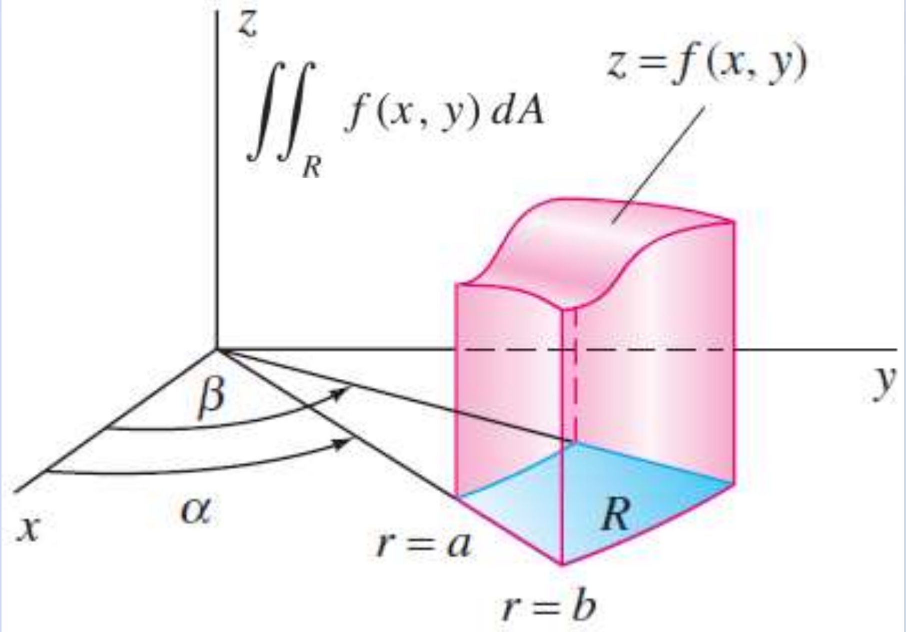
- In polar coordinates as

$$0 \leq a \leq r \leq b$$

$$\alpha \leq \theta \leq \beta$$

$$x = r\cos\theta\,,\, y = r\sin\theta$$

$$x^2 + y^2 = r^2$$



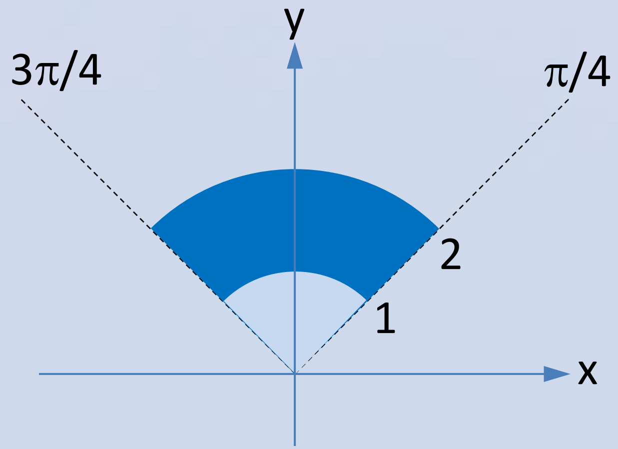
Solid region whose base is the polar rectangle R

$$\iint_R f(x, y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

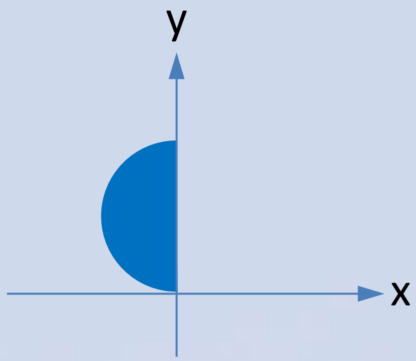
Double Integrals in Polar Coordinates
(อินทิกรัลสองชั้นในพิกัดเชิงขั้ว)

EXAMPLE จงหาค่าอินทิกรัล และวาดบริเวณพื้นที่ของอินทิกรัลต่อไปนี้

$$\int_{\pi/4}^{3\pi/4}\int_{1}^{2} r\,dr\,d\theta = \frac{3\pi}{4}$$



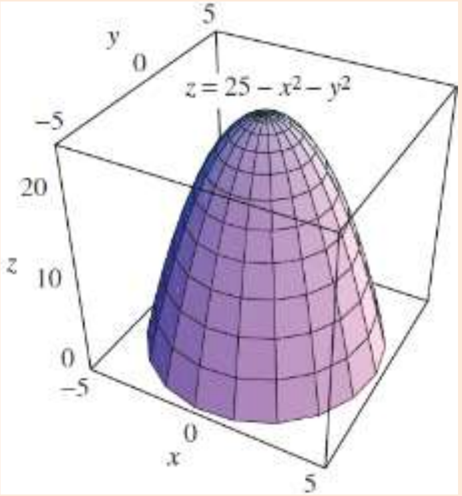
$$\int_{\pi/2}^{\pi}\int_{0}^{2\sin(\theta)} r\,dr\,d\theta = \int_{\pi/2}^{\pi} 2\sin^2(\theta)\,d\theta = \pi/2$$



<u>ตัวอย่าง</u> จงหาปริมาตร V ของรูปที่กำหนดให้โดยมีขอบเขตเป็นระนาบ xy และพาราโบลอยด์ $z = 25 - x^2 - y^2$

$$V = \iint_{x^2+y^2=25} (25 - x^2 - y^2)\mathrm{dxdy} = \int_0^{2\pi}\int_0^{5} (5^2 - r^2) r\,dr\,d\theta$$

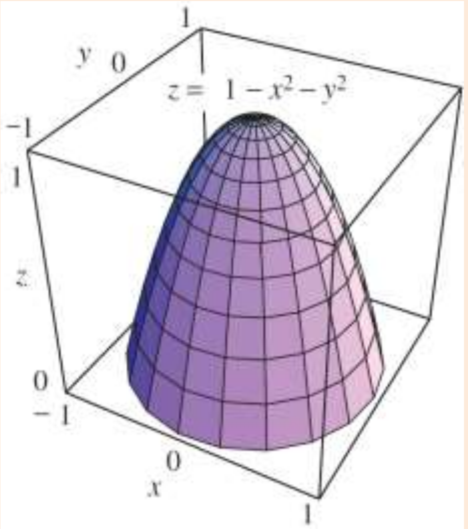
$$= \int_0^{2\pi} \left(\left[25\frac{r^2}{2} - \frac{r^4}{4} \right]_0^5 \right) d\theta = \frac{625\pi}{2}$$




```
import scipy.integrate as spi, numpy as np
spi.dblquad(lambda x, y: 25 - x**2 - y**2, -5, 5,
            lambda y: -np.sqrt(25 - y**2),
            lambda y: np.sqrt(25 - y**2))
```

ตัวอย่าง จงหาค่า $\iint_{x^2+y^2\leq 1,\, x\geq 0,\, y\geq 0}(1-x^2-y^2)dxdy$

$$= \int_0^{\pi/2}\int_0^1 (1-r^2)rdrd\theta = \frac{\pi}{2}\left[\frac{r^2}{2}-\frac{r^4}{4}\right]_0^1 = \frac{\pi}{8}$$




```
import scipy.integrate as spi
Import numpy as np
spi.dblquad(lambda x,y:1-x**2-y**2,0,1,lambda y:0,lambda y:np.sqrt(1-y**2))
```

ตัวอย่าง จงหาปริมาตร V ของ solid region ที่อยู่ภายในทรงกลม $x^2+y^2+z^2 = 4$ และ ทรงกระบอก $(x-1)^2+y^2 = 1$

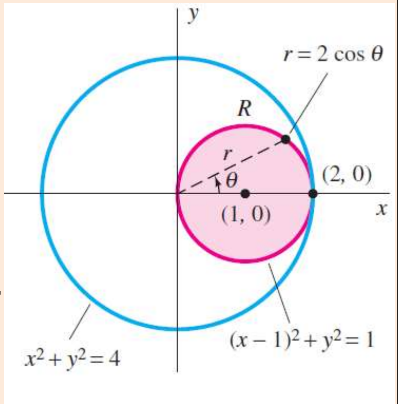
$$V = \iint_R f(x,y)dxdy = 2\int_{-\pi/2}^{\pi/2}\int_0^{2\cos(\theta)} \sqrt{4-r^2}\, rdrd\theta = 4\int_0^{\pi/2}\int_0^{2\cos(\theta)} \sqrt{4-r^2}\, rdrd\theta$$

$$= -2\int_0^{\pi/2}\int_0^{2\cos(\theta)} \sqrt{4-r^2}\, d(4-r^2)d\theta = -2\int_0^{\pi/2}\left(\frac{2}{3}(4-r^2)^{3/2}\Big]_0^{2\cos(\theta)}\right)d\theta$$

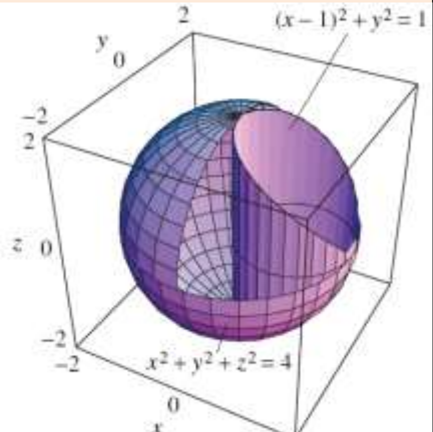
$$= -\frac{4}{3}\int_0^{\pi/2}\left(8(1-cos^2\theta)^{3/2}-8\right)d\theta = -\frac{32}{3}\int_0^{\frac{\pi}{2}}(sin^3\theta-1)\,d\theta$$

$$= \frac{32}{3}\int_1^0(1-cos^2\theta)\,d(cos\theta)+\frac{32}{3}\int_0^{\frac{\pi}{2}}d\theta = \frac{32}{3}\left(\cos\theta-\frac{cos^3\theta}{3}\Big]_0^{\frac{\pi}{2}}\right)+\frac{16\pi}{3}$$

$$= \frac{16\pi}{3}-\frac{64}{9} \approx 9.644$$

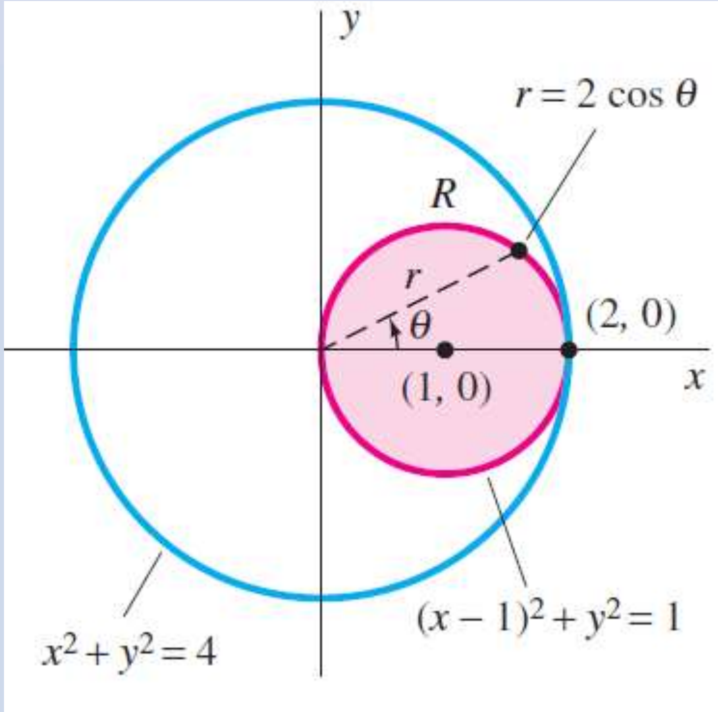


The small circle is the domain R of the integral




```
import scipy.integrate as spi
import numpy as np
result, error = spi.dblquad(
    lambda x, y: np.sqrt(4 - x**2 - y**2), # Integrand function
    -1, 1,                                  # Limits for x
    lambda y: 1 - np.sqrt(1 - y**2),        # Lower limit for y
    lambda y: 1 + np.sqrt(1 - y**2)         # Upper limit for y)
final_result = 2 * result
print("Final result:", final_result)
```

Why r = 2cos(θ) ?



The small circle is the domain R of the integral

$$(x-1)^2 + y^2 = r^2 = 1$$

We use the polar coordinate transformations:

$$x = r\cos\theta, \quad y = r\sin\theta$$

Substituting into the given equation:

$$(r\cos\theta - 1)^2 + r^2\sin^2\theta = 1$$

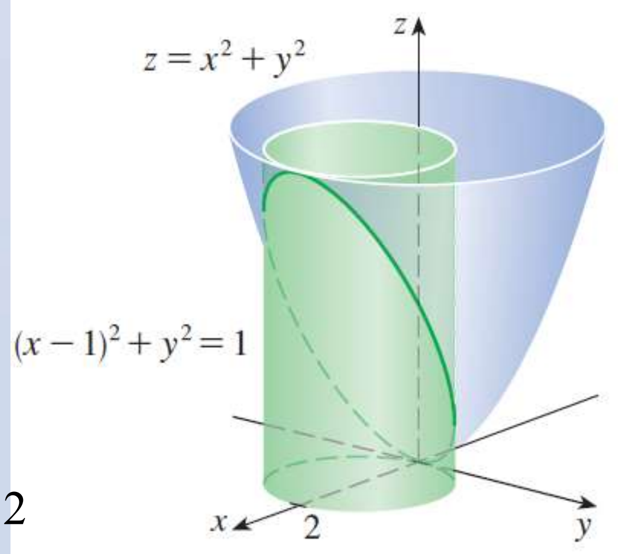
$$r^2\cos^2\theta - 2r\cos\theta + 1 + r^2\sin^2\theta = 1$$

$$r^2 - 2r\cos\theta = 0$$

$$r(r - 2\cos\theta) = 0$$

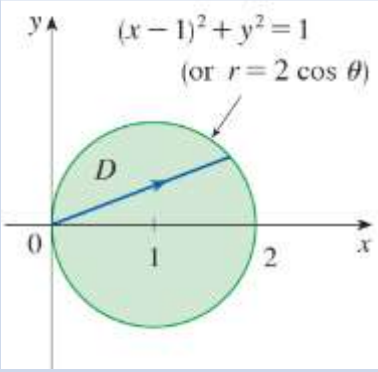
$$r = 2\cos\theta$$

ตัวอย่าง จงหาปริมาตร V ของ solid region ดังรูปซึ่งอยู่ใต้ paraboloid $z = x^2+y^2$ เหนือระนาบ xy และอยู่ภายในทรงกระบอก $x^2+y^2 = 2x$



$$V = \iint_R f(x,y)\,dxdy = \iint_R \left(x^2+y^2\right) dxdy = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos(\theta)} r^2 r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{r^4}{4} \Big|_0^{2\cos(\theta)} \right) d\theta = \int_{-\pi/2}^{\pi/2} 4\cos^4\theta\, d\theta = 4\int_{-\pi/2}^{\pi/2} \left(\frac{1+\cos(2\theta)}{2} \right)^2 d\theta$$



$$= \int_{-\pi/2}^{\pi/2} \left(1+2\cos(2\theta) + \left(\frac{1+\cos(4\theta)}{2} \right) \right) d\theta$$

$$= \theta + \sin(2\theta) + \frac{\theta}{2} + \frac{\sin(4\theta)}{8} \Big|_{-\pi/2}^{\pi/2}$$

$$= \frac{3\pi}{2}$$

<u>ตัวอย่าง</u> จงหาค่าอินทิกรัลเหล่านี้โดยแปลงตัวแปรเป็นพิกัดเชิงขั้ว (polar coordinates)

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \sqrt{x^2+y^2}\, dy\, dx = \int_{0}^{\pi}\int_{0}^{3} |r| r\, dr\, d\theta = \int_{0}^{\pi} \frac{1}{3}r^3\Big|_{0}^{3} d$$

$$= 9\int_{0}^{\pi} d\theta = 9\pi$$

```
import numpy as np
import scipy.integrate as spi
spi.dblquad(lambda x,y: np.sqrt(x**2+y**2),-3,3,0,lambda x: np.sqrt(9-x**2))
```

$$\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}} e^{x^2+y^2}\, dx\, dy = \int_{0}^{\pi/2}\int_{0}^{1} e^{r^2} r\, dr\, d\theta = \int_{0}^{\pi/2} \frac{1}{2}e^{r^2}\Big|_{0}^{1} d\theta$$

$$= \frac{1}{2}\int_{0}^{\pi/2} (e-1)\, d\theta = \frac{\pi(e-1)}{4}$$

```
import numpy as np
import scipy.integrate as spi
spi.dblquad(lambda x,y: np.exp(x**2+y**2),0,1,0,lambda y: np.sqrt(1-y**2))
```

$$\int_{-\sqrt{\pi}}^{\sqrt{\pi}}\int_{0}^{\sqrt{\pi-x^2}} \sin(x^2+y^2)\, dy\, dx = \int_{0}^{\pi}\int_{0}^{\sqrt{\pi}} (\sin r^2) r\, dr\, d\theta$$

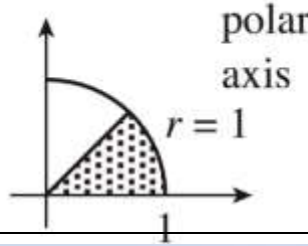
$$= \int_{0}^{\pi} -\frac{1}{2}\cos r^2\Big|_{0}^{\sqrt{\pi}} d\theta = -\frac{1}{2}\int_{0}^{\pi} (-1-1)$$

```
import numpy as np
import scipy.integrate as spi
spi.dblquad(lambda x,y: np.sin(x**2+y**2),-np.sqrt(np.pi),np.sqrt(np.pi),0,lambda x: np.sqrt(np.pi-x**2))
```

ตัวอย่าง จงหาค่าอินทิกรัลเหล่านี้โดยแปลงตัวแปรเป็นพิกัดเชิงขั้ว (polar coordinates)

$$\int_0^{\sqrt{2}/2}\int_y^{\sqrt{1-y^2}} \frac{y^2}{\sqrt{x^2+y^2}}\,dx\,dy = \int_0^{\pi/4}\int_0^1 \frac{r^2\sin^2\theta}{|r|}\,r\,dr\,d\theta = \int_0^{\pi/4}\int_0^1 r^2\sin^2\theta\,dr\,d\theta$$

$$= \int_0^{\pi/4} \frac{1}{3}r^3\sin^2\theta\Big|_0^1\,d\theta = \frac{1}{3}\int_0^{\pi/4}\sin^2\theta\,d\theta = \frac{1}{3}\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi/4} = \frac{\pi-2}{24}$$

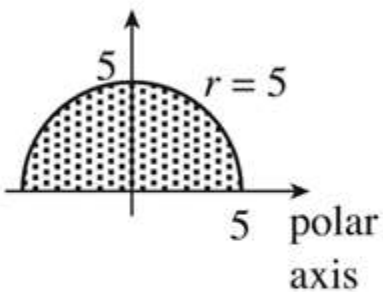



```
from sympy import symbols, sqrt, integrate
x, y = symbols('x y')
fun = y**2 / (sqrt(x**2 + y**2))
xmax = sqrt(1 - y**2) ; xmin = y
integrate(integrate(fun,(x,xmin,xmax)),(y,0,sqrt(2)/2)).evalf()
```

$$\int_{-5}^{5}\int_0^{\sqrt{25-x^2}} (4x+3y)\,dy\,dx = \int_0^{\pi}\int_0^5 (4r\cos\theta + 3r\sin\theta)r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^5 (4r^2\cos\theta + 3r^2\sin\theta)\,dr\,d\theta = \int_0^{\pi}\left(\frac{4}{3}r^3\cos\theta + r^3\sin\theta\right)\Big|_0^5\,d\theta$$

$$= \int_0^{\pi}\left(\frac{500}{3}\cos\theta + 125\sin\theta\right)d\theta = \left(\frac{500}{3}\sin\theta - 125\cos\theta\right)\Big|_0^{\pi} = 250$$




```
import numpy as np
from scipy.integrate import dblquad
def fun(y, x):# Define the function to integrate
    return 4*x + 3*y
def ymax(x):# Define the upper limit for y
    return np.sqrt(25 - x**2)
dblquad(fun, -5, 5, lambda x: 0, ymax)# Compute the double integral
```

จงหาค่าอินทิกรัลเหล่านี้โดยแปลงตัวแปรเป็นพิกัดเชิงขั้ว (polar coordinates)

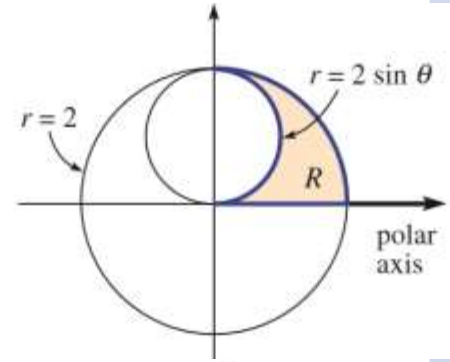
EXAMPLE Evaluate $\iint_R (3x + 4y^2)\, dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

```
from sympy import symbols, sqrt, integrate
x, y = symbols('x y')
fun = 3*x + 4*y**2
ymax = sqrt(4 - x**2) ; ymin = sqrt(1 - x**2)

s1= integrate(integrate(fun,(y,0,ymax)),(x,-2,2))
s2= integrate(integrate(fun,(y,0,ymin)),(x,-1,1))
sol = s1 - s2
print("Integral result:", sol)
```

```
import numpy as np
from scipy.integrate import dblquad
# Define the function
def f(r, theta):
    return r * (3 * np.cos(theta) + 4 * (r * np.sin(theta)) ** 2)

# Compute the double integral
sol= dblquad(f, 0, np.pi, 1, 2)
# Print result
print("Integral result:", sol)
```

ตัวอย่าง จงหาค่าอินทิกรัล $\iint_R (x+y)dA$ ทั่วขอบเขตR ดังรูป

$$\iint_R (x+y)\,dA = \int_0^{\pi/2}\int_{2\sin\theta}^{2}(r\cos\theta + r\sin\theta)\,r\,dr\,d\theta = \int_0^{\pi/2}\int_{2\sin\theta}^{2} r^2(\cos\theta+\sin\theta)\,dr\,d\theta$$



$$= \int_0^{\pi/2} \frac{1}{3}r^3(\cos\theta+\sin\theta)\Big|_{2\sin\theta}^{2}\,d\theta$$

$$= \frac{8}{3}\int_0^{\pi/2}(\cos\theta+\sin\theta-\sin^3\theta\cos\theta-\sin^4\theta)\,d\theta$$

$$= \frac{8}{3}\left(\sin\theta-\cos\theta-\frac{1}{4}\sin^4\theta+\frac{1}{4}\sin^3\theta\cos\theta-\frac{3}{8}\theta+\frac{3}{16}\sin 2\theta\right)\Big|_0^{\pi/2}$$

$$= \frac{8}{3}\left[\left(1-\frac{1}{4}-\frac{3\pi}{16}\right)-(-1)\right] = \frac{28-3\pi}{6}$$

$$\int -sin^4\theta d\theta = \int sin^3\theta\, dcos\theta = sin^3\theta\cos\theta - \int \cos\theta\, dsin^3\theta = sin^3\theta\cos\theta - 3\int sin^2\theta cos^2\theta d\theta = sin^3\theta\cos\theta - 3\int sin^2\theta(1-sin^2\theta)d\theta$$

$$= sin^3\theta\cos\theta - 3\int sin^2\theta d\theta + 3\int sin^4\theta d\theta = sin^3\theta\cos\theta - \frac{3}{2}\theta - \frac{3}{4}\sin 2\theta + 3\int sin^4\theta d\theta$$

$$4\int -sin^4\theta d\theta = sin^3\theta\cos\theta - \frac{3}{2}\theta + \frac{3}{4}\sin 2\theta$$

$$\int -sin^4\theta d\theta = \frac{sin^3\theta\cos\theta}{4} - \frac{3}{8}\theta + \frac{3}{16}\sin 2\theta$$

```
import numpy as np
from scipy.integrate import dblquad
def fun(x, y): return x + y
def xmax(y): return np.sqrt(4 - y**2)
def xmin(y): return np.sqrt(1 - (y - 1)**2)
sol, _ = dblquad(fun, 0, 2, xmin, xmax)
print("Integral result:", sol)
```

HW

39–42 Evaluate the iterated integral by converting to polar coordinates.

39. $\int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\, dy\, dx$

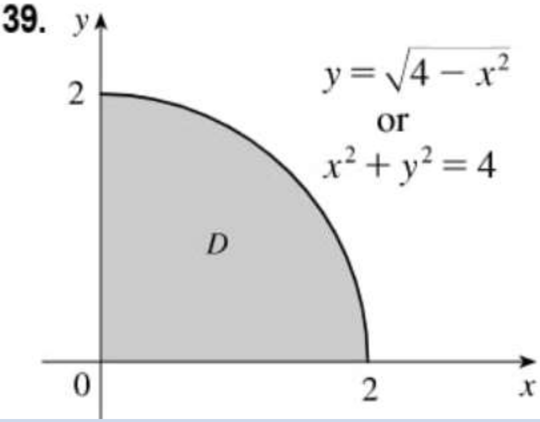
40. $\int_0^a \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} (2x + y)\, dx\, dy$

41. $\int_0^{1/2} \int_{\sqrt{3}y}^{\sqrt{1-y^2}} xy^2\, dx\, dy$

42. $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2 + y^2}\, dy\, dx$

HW

39.

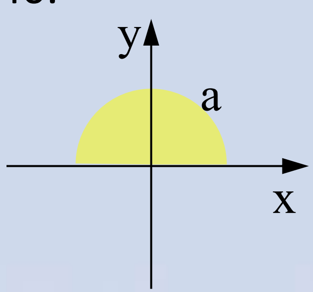


$$\int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\, dy\, dx = \int_0^{\pi/2} \int_0^2 e^{-r^2}\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} d\theta \int_0^2 r e^{-r^2}\, dr = \left[\theta\right]_0^{\pi/2} \left[-\tfrac{1}{2} e^{-r^2}\right]_0^2$$

$$= \tfrac{\pi}{2}\left[-\tfrac{1}{2}\left(e^{-4} - 1\right)\right] = \tfrac{\pi}{4}\left(1 - e^{-4}\right)$$

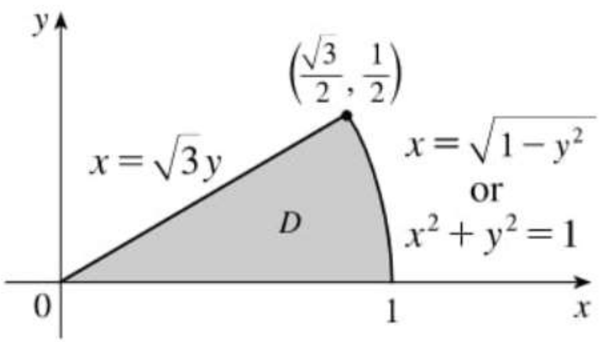
40.



$$\int_0^a \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} (2x+y)\,dx\,dy = \int_0^\pi \int_0^a r(2\cos\theta+\sin\theta)\,r\,dr\,d\theta = \frac{a^3}{3}(2\sin\theta-\cos\theta)\Big|_0^\pi = \frac{2a^3}{3}$$

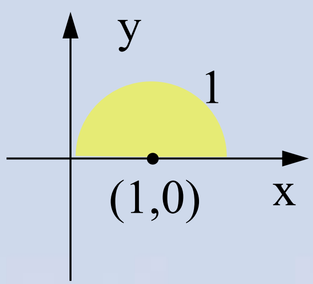
HW

41. The region D of integration is shown in the figure. In polar coordinates the line $x = \sqrt{3}\,y$ is $\theta = \pi/6$, so



$$\int_0^{1/2}\int_{\sqrt{3}\,y}^{\sqrt{1-y^2}} xy^2\,dx\,dy = \int_0^{\pi/6}\int_0^1 (r\cos\theta)(r\sin\theta)^2\,r\,dr\,d\theta$$
$$= \int_0^{\pi/6}\sin^2\theta\cos\theta\,d\theta\,\int_0^1 r^4\,dr$$
$$= \left[\tfrac{1}{3}\sin^3\theta\right]_0^{\pi/6}\left[\tfrac{1}{5}r^5\right]_0^1$$
$$= \left[\tfrac{1}{3}\left(\tfrac{1}{2}\right)^3 - 0\right]\left[\tfrac{1}{5} - 0\right] = \tfrac{1}{120}$$

42.



$$\int_0^2\int_0^{\sqrt{2x-x^2}}\sqrt{x^2+y^2}\,dydx = \int_0^{\pi/2}\int_0^{2\cos\theta}(r)rdrd\theta = \int_0^{\pi/2}\frac{8\cos^3\theta}{3}d\theta$$
$$= \frac{8}{3}\int_0^{\pi/2}\left(1-\sin^2\theta\right)d\sin\theta = \frac{8}{3}\left(\sin\theta - \frac{\sin^3\theta}{3}\right)\Bigg|_0^{\pi/2} = \frac{16}{9}$$

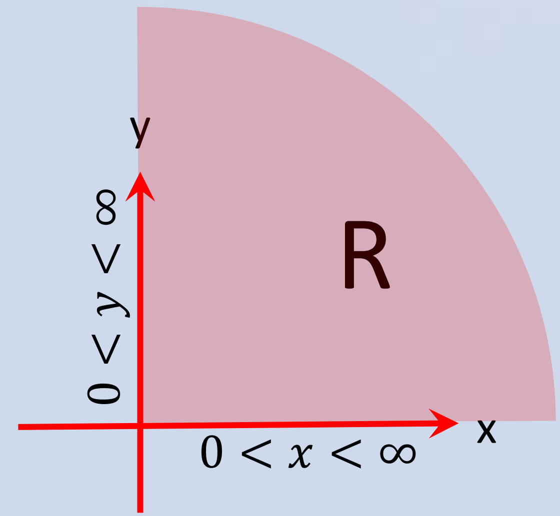
<u>ตัวอย่าง</u> จงหาค่า The improper integral $\int_0^\infty e^{-x^2}\,dx$ ซึ่งมีความสำคัญมากในทฤษฎีความน่าจะเป็น, สถิติ และอื่นๆทางด้านคณิตศาสตร์ประยุกต์ ถ้า I แทนอินทิกรัลดังกล่าว ดังนั้น $I = \int_0^\infty e^{-x^2}\,dx$ และ $I = \int_0^\infty e^{-y^2}\,dy$ ดังนั้น $I^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right) = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dxdy$

(หาค่า I^2 โดยใช้พิกัดเชิงขั้ว)

$$I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dxdy = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}\,rdrd\theta$$

$$I^2 = \int_0^{\frac{\pi}{2}} \left(\frac{1}{2}\int_0^\infty e^{-r^2}\,dr^2\right) d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2}\,d\theta = \frac{\pi}{4}$$

$$I = \frac{\sqrt{\pi}}{2}$$



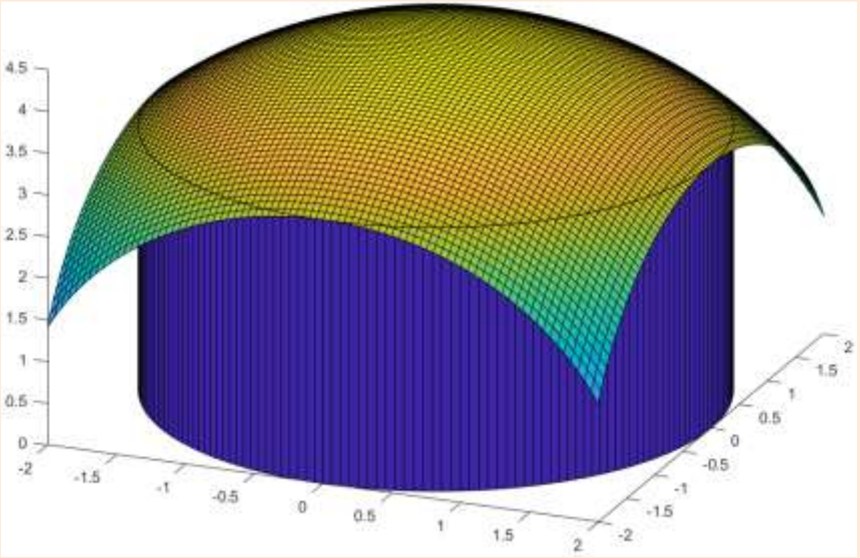

```
import numpy as np
from scipy.integrate import dblquad
def fun(x, y): return np.exp(-x**2 - y**2)
sol, _ = dblquad(fun, 0, np.inf, lambda y: 0, lambda y: np.inf)
np.sqrt(sol)
```

<u>ตัวอย่าง</u> จงหาปริมาตรภายในทรงกระบอก $x^2 + y^2 = 4$ และรูปทรงรี (ellipsoid) $2x^2 + 2y^2 + z^2 = 18$

$$V = \iint_R f(x,y)dxdy = \int_0^{2\pi}\int_0^{2} \sqrt{18 - 2r^2}\, r dr d\theta$$

$$= -\frac{1}{4}\int_0^{2\pi}\left(\frac{2}{3}(18 - 2r^2)^{\frac{3}{2}}\Big]_0^2\right)d\theta$$

$$= -\frac{\pi}{3}\left(\sqrt{1000} - \sqrt{5832}\right) \approx 46.86$$




```
import numpy as np
from scipy.integrate import dblquad
def fun(x, y): return np.sqrt(18 - 2*x**2 - 2*y**2)
def xmin(y): return -np.sqrt(4 - y**2)
def xmax(y): return np.sqrt(4 - y**2)
sol, _ = dblquad(fun, -2, 2, xmin, xmax)
print("Integral result:", sol)
```

ตัวอย่าง จงหาพื้นที่ที่ถูกปิดล้อมด้วยสมการ limaçon $r = 3 + 2\cos\theta, 0 \le \theta \le 2\pi$

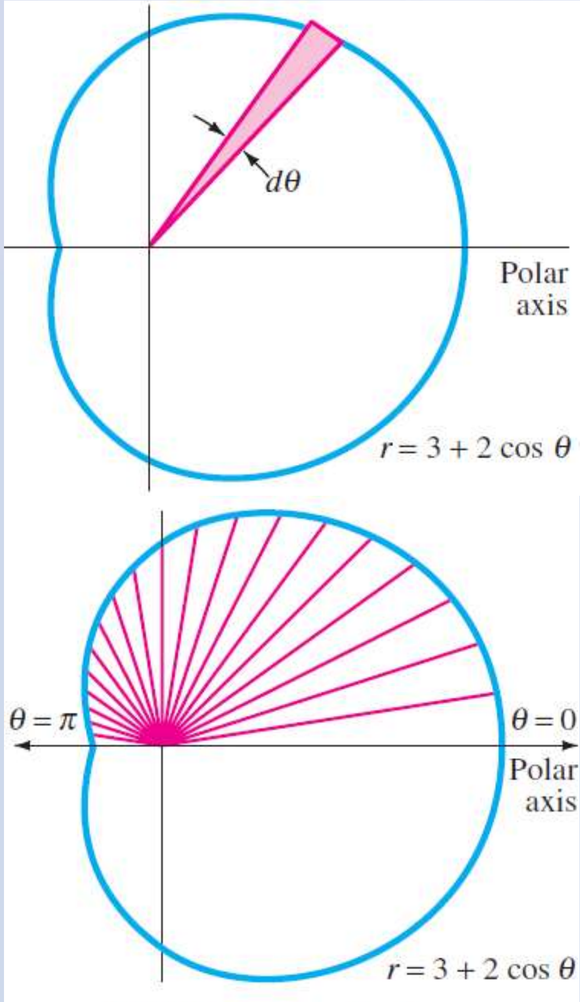
$$A = \iint r dr d\theta$$

$$= 2\int_0^{\pi}\int_0^{3+2\cos\theta} r dr d\theta = 2\int_0^{\pi} \frac{r^2}{2}\Big]_0^{3+2\cos\theta} d\theta$$

$$= \int_0^{\pi} (9 + 12\cos\theta + 4\cos^2\theta)\, d\theta$$

$$= \int_0^{\pi} \left(9 + 12\cos\theta + 4\left(\frac{\cos 2\theta + 1}{2}\right)\right) d\theta$$

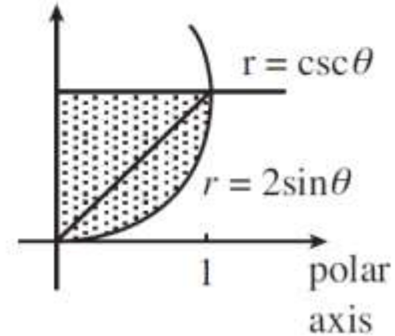
$$= (9\theta + 12\sin\theta + \sin 2\theta + 2\theta)\Big]_0^{\pi} = 11\pi$$



Infinitesimal sectors from $\theta = 0$ to $\theta = \pi$

```
import numpy as np
from scipy.integrate import dblquad
def f(r, theta): return r
def rmax(theta): return 3 + 2 * np.cos(theta)
sol, _ = dblquad(f, 0, 2 * np.pi, lambda theta: 0, rmax)
print("Integral result:", sol)
```

ตัวอย่าง จงหาค่าอินทิกรัล $\int_0^1 \int_0^{\sqrt{2y-y^2}} (1 - x^2 - y^2)\, dxdy$



$$\int_0^1 \int_0^{\sqrt{2y-y^2}} (1 - x^2 - y^2)\, dx\, dy$$

$$= \int_0^{\pi/4} \int_0^{2\sin\theta} (1 - r^2) r\, dr\, d\theta + \int_{\pi/4}^{\pi/2} \int_0^{\csc\theta} (1 - r^2) r\, dr\, d\theta$$

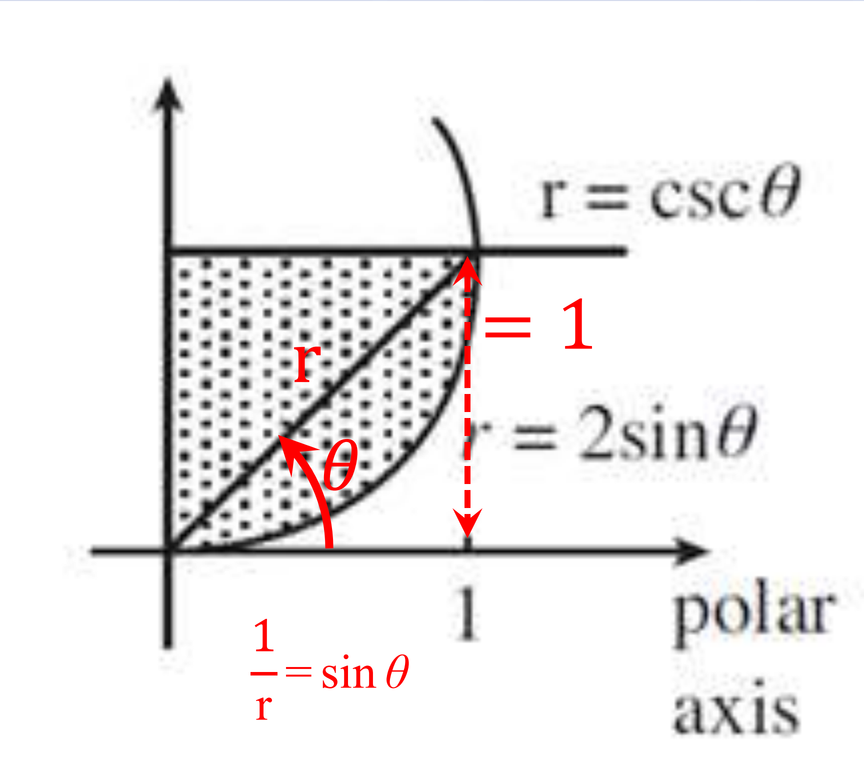
$$= \int_0^{\pi/4} \left(\frac{1}{2}r^2 - \frac{1}{4}r^4\right)\Bigg|_0^{2\sin\theta} d\theta + \int_{\pi/4}^{\pi/2} \left(\frac{1}{2}r^2 - \frac{1}{4}r^4\right)\Bigg|_0^{\csc\theta} d\theta$$

$$= \int_0^{\pi/4} (2\sin^2\theta - 4\sin^4\theta)\, d\theta + \int_{\pi/4}^{\pi/2} \left(\frac{1}{2}\csc^2\theta - \frac{1}{4}\csc^4\theta\right) d\theta$$

$$= \left[\theta - \frac{1}{2}\sin 2\theta - \left(\frac{3}{2}\theta - \sin 2\theta + \frac{1}{8}\sin 4\theta\right)\right] + \left[-\frac{1}{2}\cot\theta - \frac{1}{4}\left(-\cot\theta - \frac{1}{3}\cot^3\theta\right)\right]\Bigg|_{\pi/4}^{\pi/2}$$

$$= \left(-\frac{\pi}{8} + \frac{1}{2}\right) + \left[0 - \left(-\frac{1}{4} + \frac{1}{12}\right)\right] = \frac{16 - 3\pi}{24}$$

```
import numpy as np
from scipy.integrate import dblquad
def fun(r, theta): return r * (1 - r**2)
s1, _ = dblquad(fun, 0, np.pi/4, lambda theta: 0, lambda theta: 2 * np.sin(theta))
s2, _ = dblquad(fun, np.pi/4, np.pi/2, lambda theta: 0, lambda theta: 1 / np.sin(theta))
sol = s1 + s2
print("Integral result:", sol)
```



$$\frac{1}{r} = \sin\theta$$

$$\frac{1}{\sin\theta} = r = \csc\theta$$

$$x^2 + (y-1)^2 = r^2 = 1$$

We use the polar coordinate transformations:

$$x = r\cos\theta,\quad y = r\sin\theta$$

Substituting into the given equation:

$$r^2\cos^2\theta + (r\sin\theta - 1)^2 = 1$$

$$r^2\cos^2\theta + r^2\sin^2\theta - 2r\sin\theta + 1 = 1$$

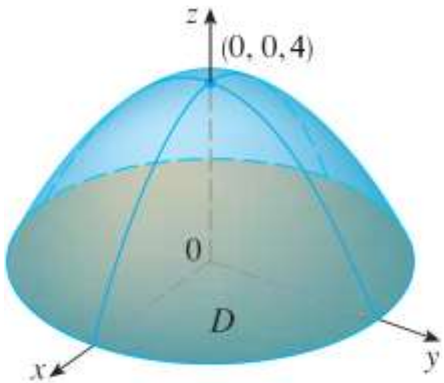
$$r^2 - 2r\sin\theta = 0$$

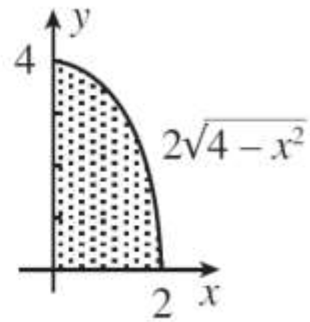
$$r(r - 2\sin\theta) = 0$$

$$r = 2\sin\theta$$

ตัวอย่าง จงหาปริมาตรของของแข็งที่ถูกปิดล้อมด้วยสมการ $z = 4 - x^2 - \dfrac{y^2}{4}$, $z = 0$

Setting $z = 0$, we have $x^2/4 + y^2/16 = 1$. Using symmetry,





$$V = 4\int_0^2 \int_0^{2\sqrt{4-x^2}} \left(4 - x^2 - \frac{1}{4}y^2\right) dy\, dx$$

$$= 4\int_0^2 \left(4y - x^2 y - \frac{1}{12}y^3\right)\Big|_0^{2\sqrt{4-x^2}} dx$$

$$= 4\int_0^2 \left[8\sqrt{4-x^2} - 2x^2\sqrt{4-x^2} - \frac{2}{3}(4-x^2)^{3/2}\right] dx$$

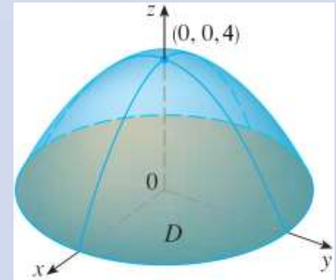
Trig substitution

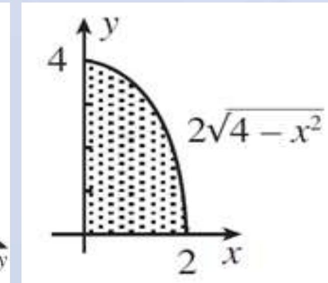
$$= 4\Big[4x\sqrt{4-x^2} + 16\sin^{-1}\frac{x}{2} - \frac{1}{4}x(2x^2-4)\sqrt{4-x^2} - 4\sin^{-1}\frac{x}{2}$$

$$+ \frac{1}{12}x(2x^2-20)\sqrt{4-x^2} - 4\sin\frac{x}{2}\Big]\Big|_0^2 = 4\left(\frac{16\pi}{2} - \frac{4\pi}{2} - \frac{4\pi}{2}\right) - (0) = 16\pi.$$

```
import sympy as sp
x, y = sp.symbols('x y')
fun = 4 - x**2 - y**2/4
ymax = sp.sqrt(16 - 4*x**2)
result = 4 * sp.integrate(sp.integrate(fun, (y, 0, ymax)), (x, 0, 2))
result
```

ตัวอย่าง จงหาปริมาตรของของแข็งที่ถูกปิดล้อมด้วยสมการ $z = 4 - x^2 - \frac{y^2}{4}$, $z = 0$





ขั้นตอน 1: หาขอบเขตของ region R

เมื่อ $z = 4 - x^2 - \frac{y^2}{4}$, $z = 0$

นี่คือสมการของวงรี (ellipse) ในระนาบ xy

ขั้นตอน 2: ตั้งปริพันธ์หาปริมาตร โดยที่ R คือบริเวณภายในวงรี $x^2 + \frac{y^2}{4} = 4$

$$V = \iint_R \left(4 - x^2 - \frac{y^2}{4}\right) dydx$$

ขั้นตอนที่ 3: แปลงเป็นพิกัดเชิงขั้ว (scaled polar coordinates)

ใช้:

ขอบเขตวงรี $x^2 + \frac{y^2}{4} = 4$ หรือ $r^2\cos^2\theta + \frac{4r^2\sin^2\theta}{4} = 4$ กลายเป็น: $r^2 = 4$ หรือ $r = 2$ ฟังก์ชันภายใต้ปริพันธ์จะกลายเป็น :

$$\int_0^{2\pi}\int_0^{2}(4 - r^2)2rdrd\theta = 2\pi\left(4r^2 - \frac{r^4}{2}\right)\Bigg|_0^2 = 16\pi$$