

# ADVANCED ENGINEERING MATHEMATICS

Chapter 7

Lecture 1 - Vector

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# Outline

1. Vectors in 2-Space (2D)
2. Vectors in 3-Space (3D)
3. Dot Product
4. Cross Product
5. Lines and Planes in 3-Space
6. Vector Spaces
7. Gram–Schmidt Orthogonalization Process

เว็บไซต์ : [www.symbolab.com](http://www.symbolab.com)

# Derivatives Review:

## □ Basic Derivatives:

- Compute the derivatives of  $f(x)=3x^2+2x-1$
- Find the derivatives of  $g(x)=\sin(x)$

$$f'(x)=6x+2$$

$$g'(x)=\cos(x)$$

## □ Product and Quotient Rule:

- Apply the product rule to find  $h'(x)$   
if  $h(x)=(2x+1)(3x-4)$

$$h'(x)=2(3x-4)+(2x+1)3$$

- Use the quotient rule to find  $\frac{d}{dx}\left(\frac{x^2+1}{x-2}\right)$

$$\frac{2x(x-2)-(x^2+1)}{(x-2)^2}$$

## □ Chain Rule:

- Compute the derivatives of  $y=\sqrt{2x+1}$   
using the chain rule
- Find  $\frac{d}{dx}(\cos(3x^2))$

$$y'=\frac{1}{2\sqrt{2x+1}}(2)=\frac{1}{\sqrt{2x+1}}$$

$$\frac{d}{dx}(\cos(3x^2))=-6x\sin(3x^2)$$

# Integral Review:

## □ Basic:

- Evaluate  $\int (4x^3 + 2x^2 - 3) dx$

- Find  $\int e^x dx = e^x + C$

$$= x^4 + \frac{2x^3}{3} - 3x + C$$

## □ Definite Integrals:

- Calculate  $\int_0^1 (2x+1) dx = 2$

- Evaluate  $\int_{-\pi}^{\pi} \sin(x) dx = 0$

## □ Integration by Parts:

- Use integration by parts to evaluate  $\int x \cos(x) dx = x \sin(x) + \cos(x) + C$

- Evaluate  $\int e^x \sin(x) dx = \frac{-e^x \cos(x) + e^x \sin(x)}{2} + C$

## □ Chain Rule:

- Make the substitution  $u=2x+1$  and evaluate  $\int \frac{1}{2x+1} dx = \frac{1}{2} \ln(2x+1) + C$

- Use substitution to find  $\int x^2 \sqrt{x^3+1} dx$

$$= \frac{2(x^3+1)^{\frac{3}{2}}}{9} + C$$

# Vectors in 2-Space

- A **scalar** is a real number or quantity that has a **magnitude**, such as length and temperature
- A **vector** has both **magnitude** and **direction** and it can be represented by a boldface symbol or a symbol under an arrow,  $\mathbf{v}$  or  $\overrightarrow{AB}$

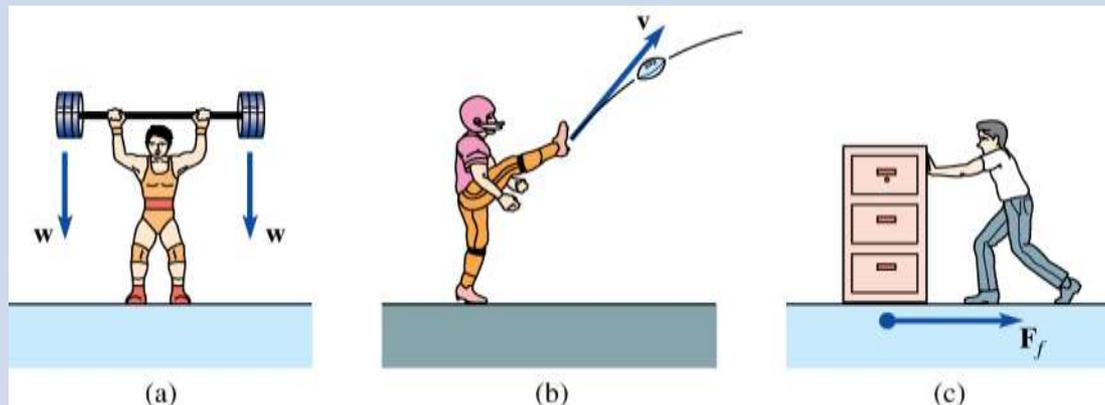


Figure 07.1.1: Examples of vector quantities

## Vectors in 2-Space (cont'd.)

- Characteristics of vectors
  - The vector  $\overrightarrow{AB}$  has an initial point at  $A$  and a terminal point at  $B$
  - **Equal** vectors have the same magnitude and direction
  - The **negative** of a vector has the same magnitude and opposite direction

## Vectors in 2-Space (cont'd.)

- If  $k \neq 0$  is a scalar, the **scalar multiple** of a vector  $k \overrightarrow{AB}$  is a vector that is  $|k|$  times  $\overrightarrow{AB}$
- Two vectors are **parallel** if they are nonzero scalar multiples of each other

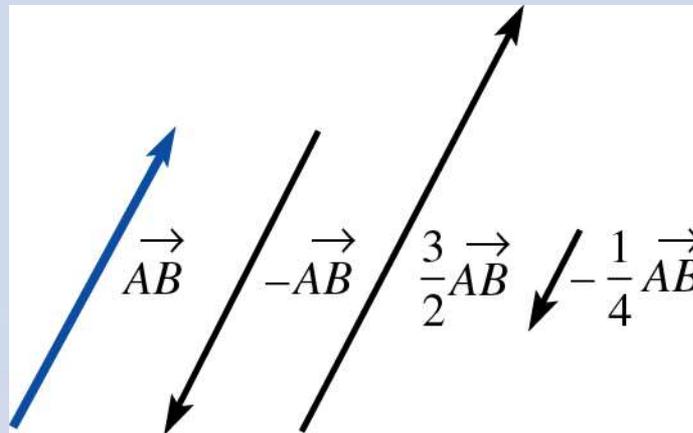
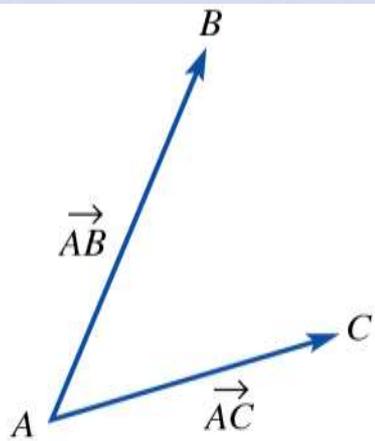
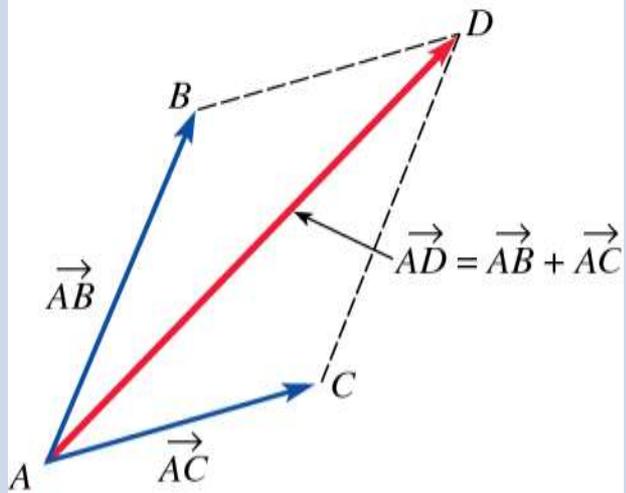


Figure: Parallel vectors

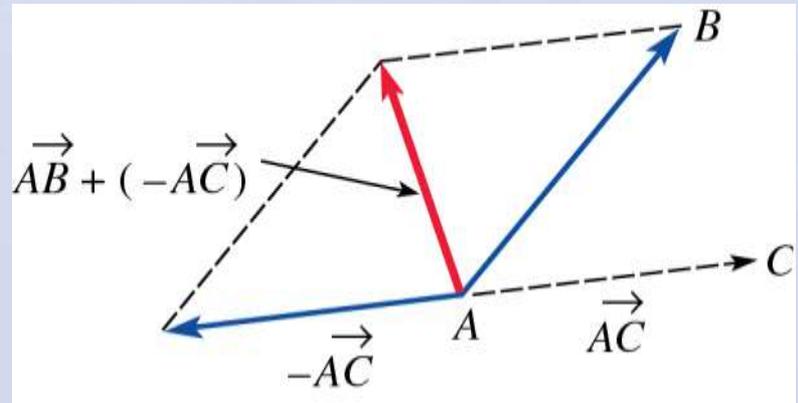


(a)

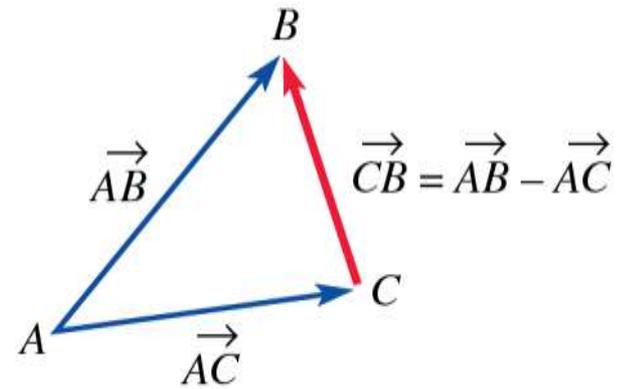


(b)

Figure: Vector  $\vec{AD}$  is the sum of  $\vec{AB}$  and  $\vec{AC}$



(a)

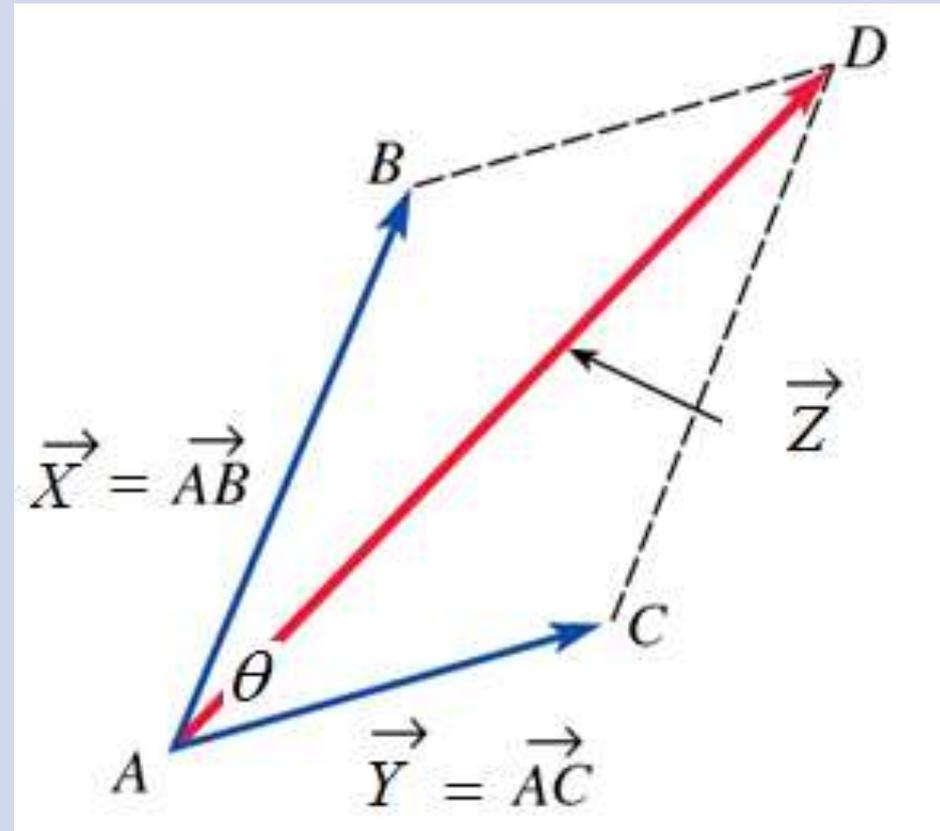


(b)

Figure: Vector  $\vec{CB}$  is the difference of  $\vec{AB}$  and  $\vec{AC}$

## Addition and Subtraction of 2-space vectors

# Addition of 2-space vectors

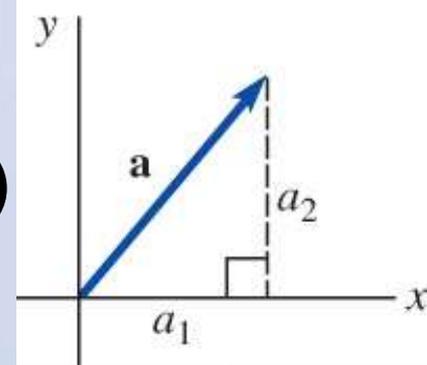


## Law of Cosines

$$|\vec{Z}| = \sqrt{|\vec{X}|^2 + |\vec{Y}|^2 + 2|\vec{X}||\vec{Y}|\cos(\theta)}$$

## Vectors in 2-Space (cont'd.)

$$\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} = \langle 2, -3 \rangle$$



- A vector  $\mathbf{a} = \langle a_1, a_2 \rangle$  is an ordered pair of real numbers where  $a_1$  and  $a_2$  are the **components** of the vector
  - Addition and subtraction of vectors, multiplication of vectors by scalars, and so on, are defined in terms of components

### Definition 7.1.1 Addition, Scalar Multiplication, Equality

Let  $\mathbf{a} = \langle a_1, a_2 \rangle$  and  $\mathbf{b} = \langle b_1, b_2 \rangle$  be vectors in  $R^2$ .

- (i) Addition:  $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$  (1)
- (ii) Scalar multiplication:  $k\mathbf{a} = \langle ka_1, ka_2 \rangle$  (2)
- (iii) Equality:  $\mathbf{a} = \mathbf{b}$  if and only if  $a_1 = b_1, a_2 = b_2$  (3)

# Vectors in 2-Space (cont'd.)

- The component definition of a vector can be used to verify the following properties of vectors

## Theorem 7.1.1 Properties of Vectors

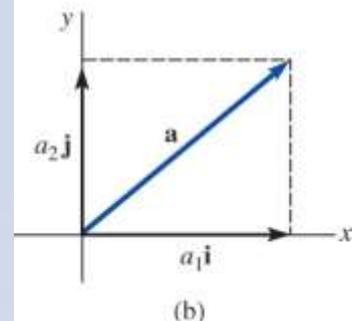
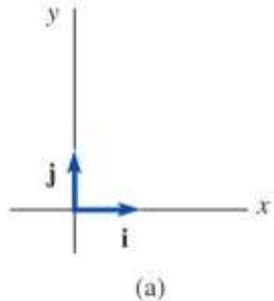
(i)	$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$	← commutative law	สลับที่
(ii)	$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$	← associative law	เปลี่ยนหมู่
(iii)	$\mathbf{a} + \mathbf{0} = \mathbf{a}$	← additive identity	เอกลักษณ์การบวก
(iv)	$\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$	← additive inverse	ผกผันการบวก
(v)	$k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$ , $k$ a scalar		
(vi)	$(k_1 + k_2)\mathbf{a} = k_1\mathbf{a} + k_2\mathbf{a}$ , $k_1$ and $k_2$ scalars		
(vii)	$k_1(k_2\mathbf{a}) = (k_1k_2)\mathbf{a}$ , $k_1$ and $k_2$ scalars		
(viii)	$1\mathbf{a} = \mathbf{a}$		
(ix)	$0\mathbf{a} = \mathbf{0}$	← zero vector	

## Vectors in 2-Space (cont'd.)

- The **magnitude**, **length**, or **norm** of a vector  $\mathbf{a}$  is denoted by  $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2}$
- A vector  $\mathbf{u}$  with magnitude 1 is a **unit vector**
  - $\mathbf{u} = (1 / \|\mathbf{a}\|)\mathbf{a}$  is the normalization of  $\mathbf{a}$
  - The unit vectors  $\mathbf{i} = \langle 1, 0 \rangle$  and  $\mathbf{j} = \langle 0, 1 \rangle$  are the standard basis for two-dimensional vectors

$$\mathbf{a} = \langle a_1, a_2 \rangle = a_1\mathbf{i} + a_2\mathbf{j}$$

where  $a_1$  and  $a_2$  are **horizontal** and **vertical components** of  $\mathbf{a}$ , respectively



# Vectors in 2-Space (cont'd.)

## EXAMPLE

### Vector Operations Using $\mathbf{i}$ and $\mathbf{j}$

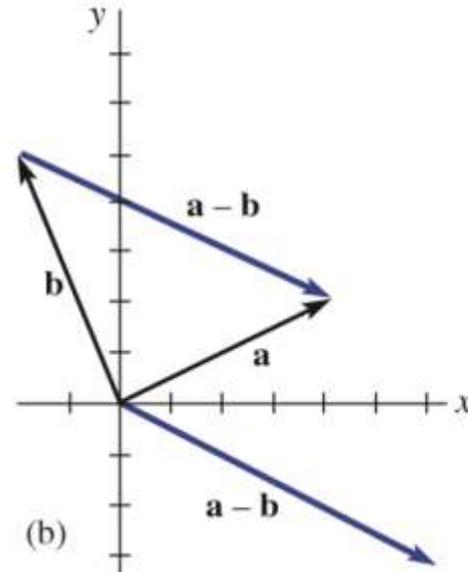
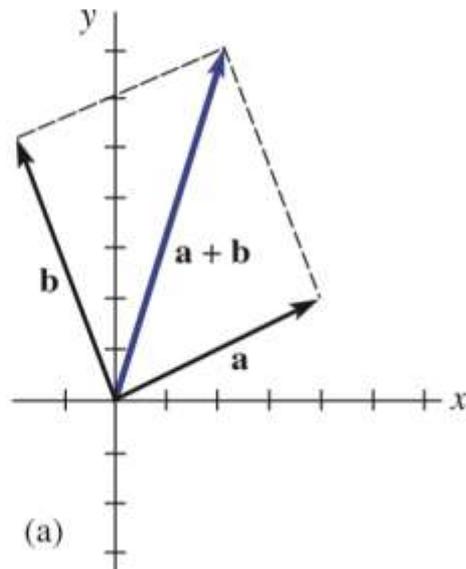
- (a)  $\langle 4, 7 \rangle = 4\mathbf{i} + 7\mathbf{j}$
- (b)  $(2\mathbf{i} - 5\mathbf{j}) + (8\mathbf{i} + 13\mathbf{j}) = 10\mathbf{i} + 8\mathbf{j}$
- (c)  $\|\mathbf{i} + \mathbf{j}\| = \sqrt{2}$
- (d)  $10(3\mathbf{i} - \mathbf{j}) = 30\mathbf{i} - 10\mathbf{j}$
- (e)  $\mathbf{a} = 6\mathbf{i} + 4\mathbf{j}$ ,  $\mathbf{b} = 9\mathbf{i} + 6\mathbf{j}$  are parallel, since  $\mathbf{b}$  is a scalar multiple of  $\mathbf{a}$ .  $\mathbf{b} = \frac{3}{2}\mathbf{a}$ .

## EXAMPLE

### Graphs of Vector Sum/Vector Difference

Let  $\mathbf{a} = 4\mathbf{i} + 2\mathbf{j}$  and  $\mathbf{b} = -2\mathbf{i} + 5\mathbf{j}$ . Graph  $\mathbf{a} + \mathbf{b}$  and  $\mathbf{a} - \mathbf{b}$ .

**SOLUTION** The graphs of  $\mathbf{a} + \mathbf{b} = 2\mathbf{i} + 7\mathbf{j}$  and  $\mathbf{a} - \mathbf{b} = 6\mathbf{i} - 3\mathbf{j}$  are given



# Vectors in 3-Space

- In three dimensions, or **3-space**, a rectangular coordinate system is constructed with three mutually orthogonal axes

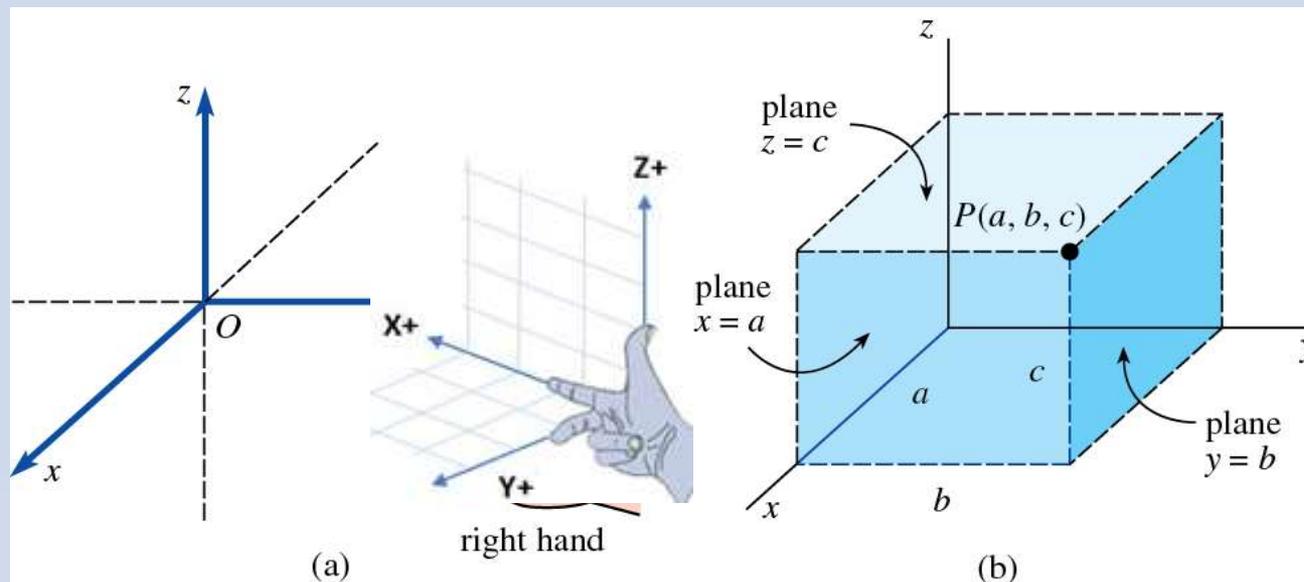


Figure: Rectangular coordinates in 3-space

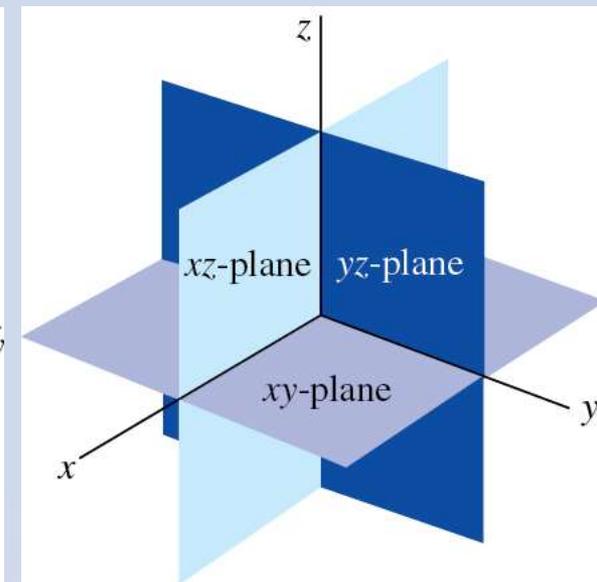


Figure: Octants

## Vectors in 3-Space (cont'd.)

- The **distance** between two points

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

จงหาระยะทาง (d) ระหว่างจุด (2,-3,6) และ (-1,-7,4)

$$d = \sqrt{(2 - (-1))^2 + (-3 - (-7))^2 + (6 - 4)^2} = \sqrt{9 + 16 + 4} = \sqrt{29}$$

- The coordinates of the **midpoint** of a line segment between two points

$$\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

# Vectors in 3-Space (cont'd.)

- A vector  $\mathbf{a}$  in 3-space is any ordered triple of real numbers

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

## Definition 7.2.1 Component Definitions in 3-Space

Let  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  and  $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$  be vectors in  $R^3$ .

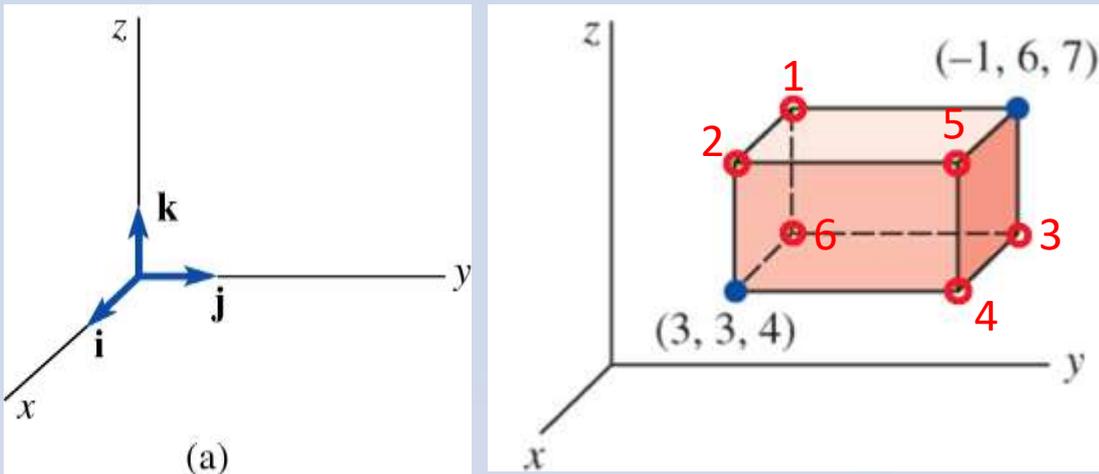
- (i) Addition:  $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$
- (ii) Scalar multiplication:  $k\mathbf{a} = \langle ka_1, ka_2, ka_3 \rangle$
- (iii) Equality:  $\mathbf{a} = \mathbf{b}$  if and only if  $a_1 = b_1, a_2 = b_2, a_3 = b_3$
- (iv) Negative:  $-\mathbf{b} = (-1)\mathbf{b} = \langle -b_1, -b_2, -b_3 \rangle$
- (v) Subtraction:  $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$
- (vi) Zero vector:  $\mathbf{0} = \langle 0, 0, 0 \rangle$
- (vii) Magnitude:  $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

# Vectors in 3-Space (cont'd.)

- Any vector  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  can be expressed as a linear combination of the unit vectors

$$\mathbf{i} = \langle 1, 0, 0 \rangle \quad \mathbf{j} = \langle 0, 1, 0 \rangle \quad \mathbf{k} = \langle 0, 0, 1 \rangle$$

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$



จุด 1 = (-1, 3, 7)

จุด 2 = (3, 3, 7)

จุด 3 = (-1, 6, 4)

จุด 4 = (3, 6, 4)

จุด 5, 6 = ?

(3, 6, 7), (-1, 3, 4)

Figure 07.2.8a:  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$  form a basis for  $\mathbb{R}^3$

# Dot Product

- In 2-space,  $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2$
- In 3-space,
- In n-space,

## Theorem 7.3.1 Properties of the Dot Product

- (i)  $\mathbf{a} \cdot \mathbf{b} = 0$  if  $\mathbf{a} = \mathbf{0}$  or  $\mathbf{b} = \mathbf{0}$
- (ii)  $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$  ← commutative law
- (iii)  $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$  ← distributive law
- (iv)  $\mathbf{a} \cdot (k\mathbf{b}) = (k\mathbf{a}) \cdot \mathbf{b} = k(\mathbf{a} \cdot \mathbf{b})$ ,  $k$  a scalar
- (v)  $\mathbf{a} \cdot \mathbf{a} \geq 0$
- (vi)  $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$

- Alternative form of the dot product  $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta$

$$\mathbf{a} = -3\mathbf{i} - \mathbf{j} + 4\mathbf{k}, \mathbf{b} = 2\mathbf{i} + 14\mathbf{j} + 5\mathbf{k}, \quad \mathbf{a} \cdot \mathbf{b} =$$

## Dot Product (cont'd.)

- 2 vectors (**a** and **b**) are *orthogonal* if  $\mathbf{a} \cdot \mathbf{b} = 0$
- The angle between two vectors is given by

$$\cos \theta = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\|\mathbf{a}\| \|\mathbf{b}\|}$$

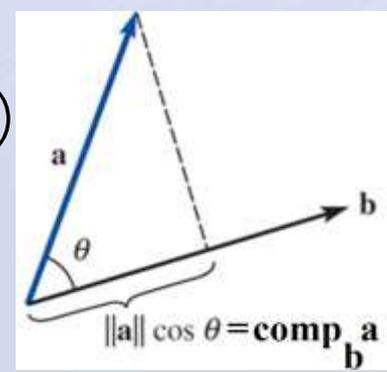
$$\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}, \mathbf{b} = -\mathbf{i} + 5\mathbf{j} + \mathbf{k}$$

$$\|\mathbf{a}\| = \sqrt{14}, \|\mathbf{b}\| = \sqrt{27}, \mathbf{a} \cdot \mathbf{b} = 14$$

$$\cos \theta = \frac{14}{\sqrt{14}\sqrt{27}} \Rightarrow \theta \approx 43.9^\circ$$

# Component of $\mathbf{a}$ on $\mathbf{b}$ (ปริมาณสเกลาร์)

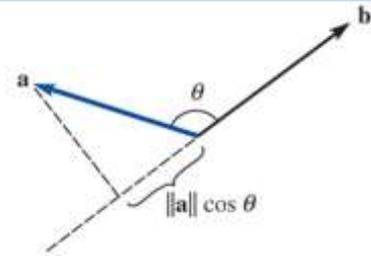
- The component of  $\mathbf{a}$  on  $\mathbf{b}$  คือ  $\text{comp}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|}$



$$\|\mathbf{b}\| = \sqrt{6}, \quad \|\mathbf{a}\| = \sqrt{29}$$

$$\text{comp}_{\mathbf{b}}\mathbf{a} = (2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) \cdot \frac{1}{\sqrt{6}} (\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = -\frac{3}{\sqrt{6}}$$

$$\text{comp}_{\mathbf{a}}\mathbf{b} = (\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \cdot \frac{1}{\sqrt{29}} (2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) = -\frac{3}{\sqrt{29}}$$



# Component of $\mathbf{a}$ on $\mathbf{b}$

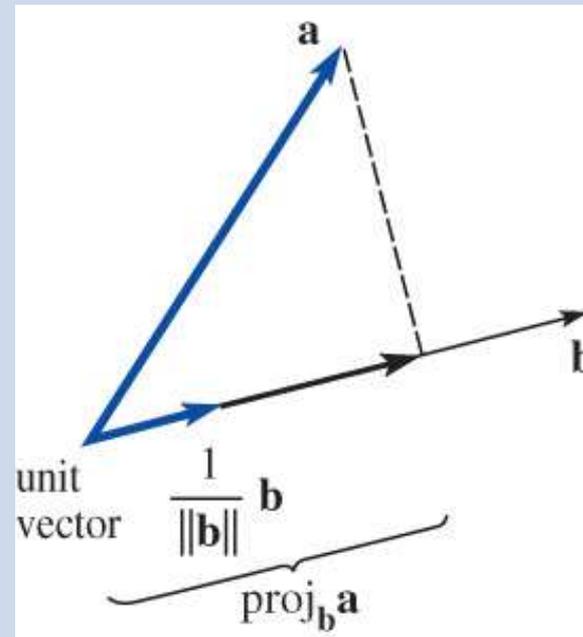
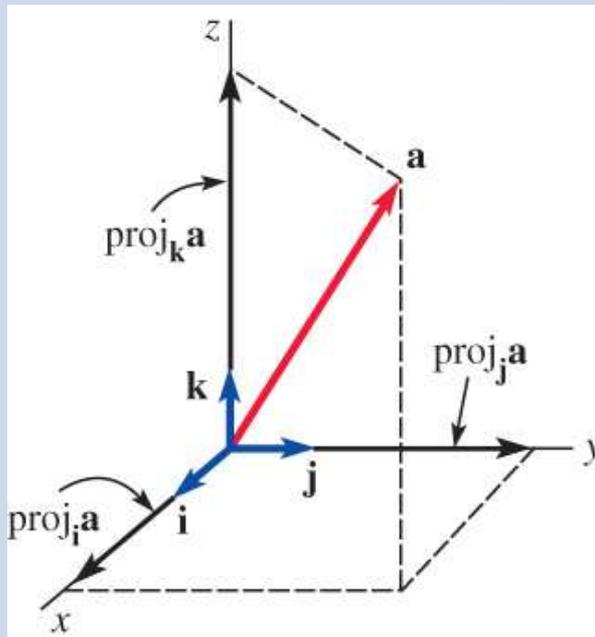
- The component of  $\mathbf{a}$  on  $\mathbf{i}$   $\Rightarrow \text{comp}_{\mathbf{i}}\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{i}}{\|\mathbf{i}\|} = \mathbf{a} \cdot \mathbf{i}$
- The component of  $\mathbf{a}$  on  $\mathbf{j}$   $\Rightarrow \text{comp}_{\mathbf{j}}\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{j}}{\|\mathbf{j}\|} = \mathbf{a} \cdot \mathbf{j}$
- The component of  $\mathbf{a}$  on  $\mathbf{k}$   $\Rightarrow \text{comp}_{\mathbf{k}}\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{k}}{\|\mathbf{k}\|} = \mathbf{a} \cdot \mathbf{k}$

Component of  $\mathbf{a}$  on a vector  $\mathbf{b}$ , we dot  $\mathbf{a}$  with a unit vector in the direction of  $\mathbf{b}$

# Projection of **a** on **b** (ปริมาณเวกเตอร์)

เวกเตอร์หน่วยตามแนว **b**

$$\text{proj}_b a = (\text{comp}_b a) \left( \frac{\mathbf{b}}{\|\mathbf{b}\|} \right) = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2} \right) \left( \frac{\mathbf{b}}{\|\mathbf{b}\|} \right) = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \right) \mathbf{b}$$



# Projection of $\mathbf{a}$ on $\mathbf{b}$

$$\text{proj}_{\mathbf{b}}\mathbf{a} = (\text{comp}_{\mathbf{b}}\mathbf{a}) \left( \frac{\mathbf{b}}{\|\mathbf{b}\|} \right) = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|} \right) \left( \frac{\mathbf{b}}{\|\mathbf{b}\|} \right) = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \right) \mathbf{b}$$

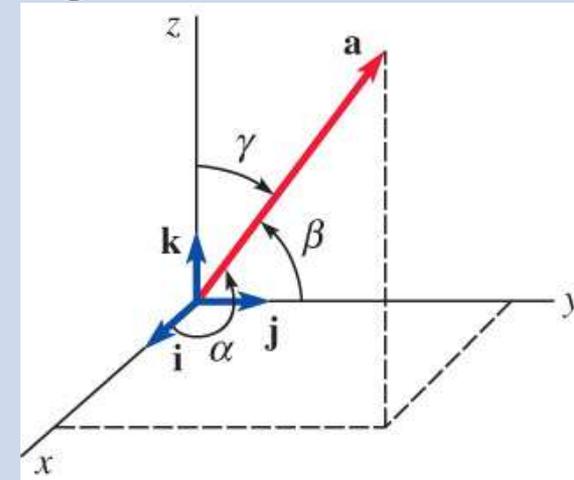
Find the projection of  $\mathbf{a} = 4\mathbf{i} + \mathbf{j}$  onto the vector  $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j}$ . Graph.

# Dot Product (Direction Cosines)

ถ้า  $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ , มุม  $\alpha, \beta, \gamma$  ระหว่างเวกเตอร์  $\mathbf{a}$  และเวกเตอร์  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  จะถูกเรียกว่า “*Direction Angles*”

$$\cos \alpha = \frac{\mathbf{a} \cdot \mathbf{i}}{\|\mathbf{a}\| \|\mathbf{i}\|}, \cos \beta = \frac{\mathbf{a} \cdot \mathbf{j}}{\|\mathbf{a}\| \|\mathbf{j}\|}, \cos \gamma = \frac{\mathbf{a} \cdot \mathbf{k}}{\|\mathbf{a}\| \|\mathbf{k}\|}$$

$$\cos \alpha = \frac{a_1}{\|\mathbf{a}\|}, \cos \beta = \frac{a_2}{\|\mathbf{a}\|}, \cos \gamma = \frac{a_3}{\|\mathbf{a}\|}$$



$\cos \alpha, \cos \beta, \cos \gamma$  are called "*Direction Cosines*"

$$\mathbf{a} = 2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$$

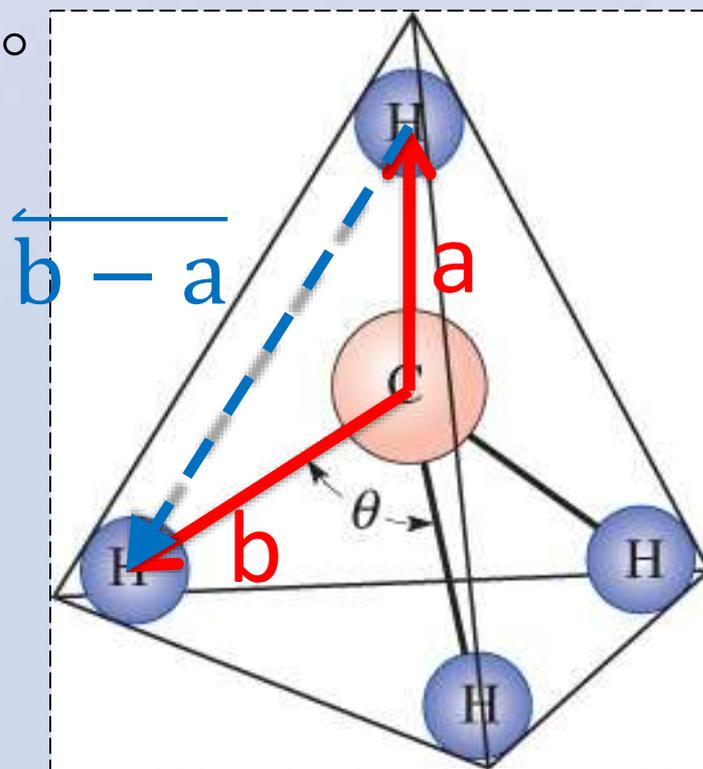
$$\alpha = \cos^{-1}\left(\frac{2}{\sqrt{45}}\right) \approx 72.7^\circ \quad \beta = \cos^{-1}\left(\frac{5}{\sqrt{45}}\right) \approx 41.8^\circ \quad \gamma = \cos^{-1}\left(\frac{4}{\sqrt{45}}\right) \approx 53.4^\circ$$

## ตัวอย่าง

สมมุติให้ระยะ  $C - H$  เท่ากับ  $1.1 \text{ \AA}$  ,  $\theta = 109.5^\circ$

จงหาระยะ  $H - H = ?$

$$\text{ระยะ } H - H = \|b - a\|$$



$$\|b - a\| = \sqrt{(b - a) \cdot (b - a)} = \sqrt{b \cdot b - 2a \cdot b + a \cdot a}$$

$$= \sqrt{\|b\|^2 + \|a\|^2 - 2\|a\|\|b\|\cos\theta} = \sqrt{(1.1)^2 + (1.1)^2 - 2(1.1)(1.1)\cos(109.5^\circ)}$$

$$= \sqrt{1.21 + 1.21 - 2.42(-0.333807)} \approx 1.80 \text{ angstroms}$$

# Cross Product

- **Determinants** (ทบคูณ)

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \quad \begin{vmatrix} -4 & -2 \\ 5 & 3 \end{vmatrix} = (-4)(3) - (5)(-2) = -2$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

- **Cross Product** of 2 vectors is given by

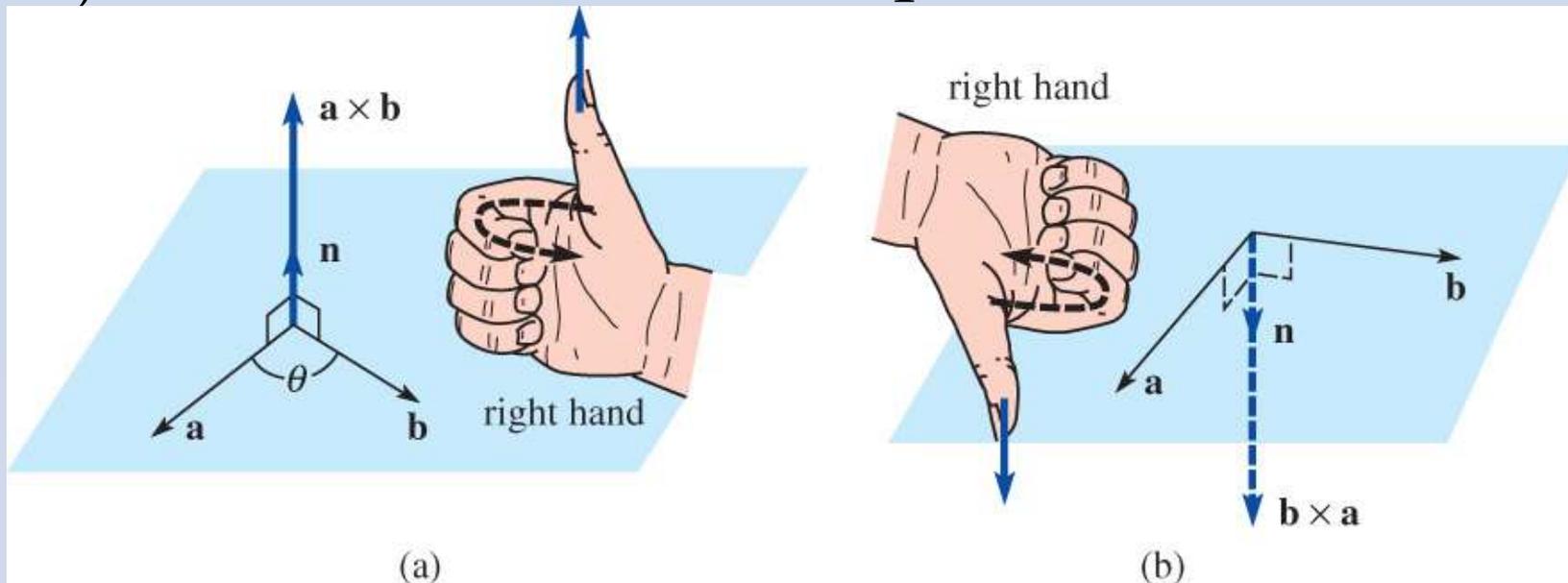
$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}(a_2 b_3 - a_3 b_2) - \mathbf{j}(a_1 b_3 - a_3 b_1) + \mathbf{k}(a_1 b_2 - a_2 b_1)$$

# Cross Product (cont'd.)

- $\mathbf{a} \times \mathbf{b}$  is *orthogonal* to the plane containing  $\mathbf{a}$  and  $\mathbf{b}$
- The magnitude of the cross product is given by

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin \theta$$

- 2 nonzero vectors are parallel if  $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin 0^\circ = 0$
- $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$  if  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  are coplanar

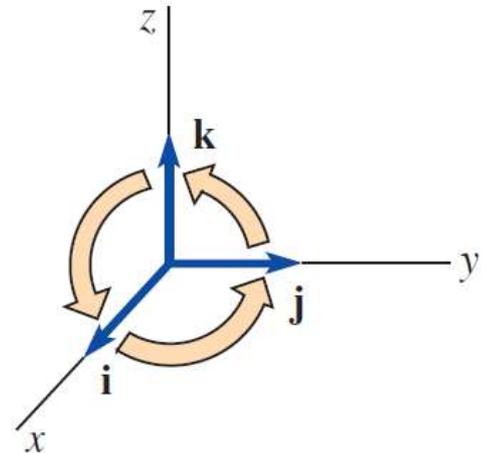


# Cross Product (cont'd.)

## Theorem 7.4.1

## Properties of the Cross Product

- (i)  $\mathbf{a} \times \mathbf{b} = \mathbf{0}$  if  $\mathbf{a} = \mathbf{0}$  or  $\mathbf{b} = \mathbf{0}$  or  $\mathbf{a}$  parallel  $\mathbf{b}$
- (ii)  $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
- (iii)  $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$
- (iv)  $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c})$
- (v)  $\mathbf{a} \times (k\mathbf{b}) = (k\mathbf{a}) \times \mathbf{b} = k(\mathbf{a} \times \mathbf{b})$ ,  $k$  a scalar
- (vi)  $\mathbf{a} \times \mathbf{a} = \mathbf{0}$
- (vii)  $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$
- (viii)  $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$



A mnemonic for cross products involving  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$

# Lines and Planes in 3-Space

How to find various equations of lines and planes in 3-space.

• The **vector equation** for a line  $\mathcal{L}_a$  is  $\mathbf{r} = \mathbf{r}_2 + t\mathbf{a}$

—  $\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2$  คือเวกเตอร์จากจุดกำเนิดชี้ไปที่เส้นตรง

— สเกลาร์  $t$  คือพารามิเตอร์

— เวกเตอร์  $\mathbf{a}$  คือเวกเตอร์ทิศทาง  
(มีทิศทางเดียวกับเวกเตอร์  $\mathbf{r} - \mathbf{r}_2$ )

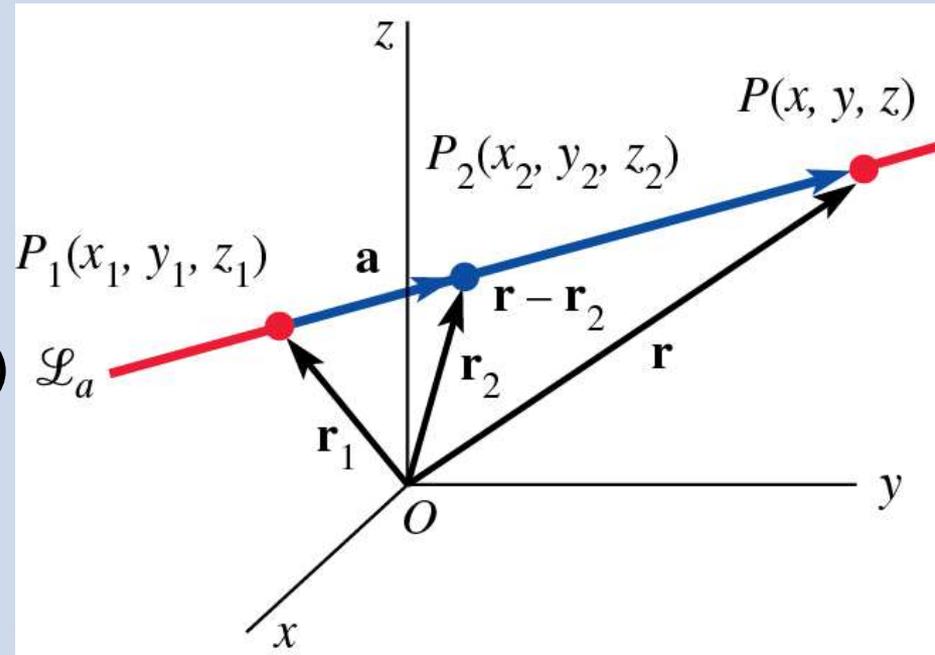
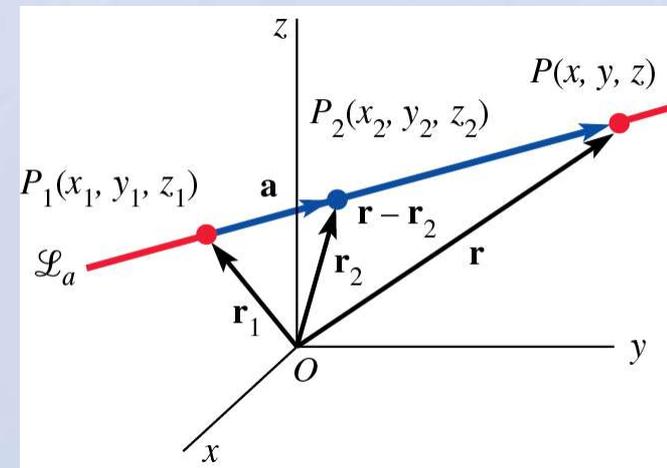


Figure 07.5.1: Line through distinct points in 3-space

# Example of Lines in 3-Space



- จงหาสมการเวกเตอร์สำหรับเส้นตรง(vector equation for a line  $\mathcal{L}_a$ ) ที่ผ่าน  $(2,-1,8)$  และ  $(5,6,-3)$

$$a = \langle 2-5, -1-6, 8-(-3) \rangle = \langle -3, -7, 11 \rangle \quad \text{หรือ} \quad a = \langle 5-2, 6-(-1), -3-8 \rangle = \langle 3, 7, -11 \rangle$$

ดังนั้น สมการเวกเตอร์ มีค่าเท่ากับ

$$\langle x, y, z \rangle = r_2 + t \cdot a = \langle 2, -1, 8 \rangle + t \langle -3, -7, 11 \rangle$$

$$\langle x, y, z \rangle = r_2 + t \cdot a = \langle 5, 6, -3 \rangle + t \langle 3, 7, -11 \rangle$$

# Lines and Planes in 3-Space (cont'd.)

- The vector equation for a plane  $\mathcal{P}$  is  $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_1) = 0$ 
  - Plane passes through a given point and has a specified normal vector  $\mathbf{n}$
  - $\mathbf{r}_1$  and  $\mathbf{r}$  are vectors from the origin  $(0,0)$  to points  $P_1, P$  on the plane  $\mathcal{P}$

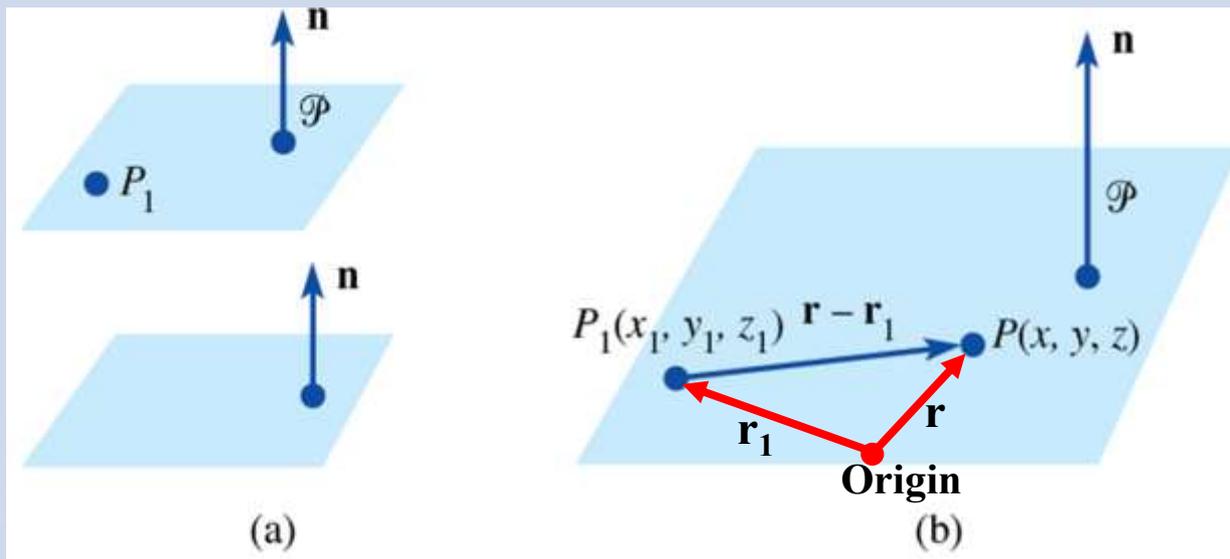


Figure: Vector  $\mathbf{n}$  is perpendicular to a plane

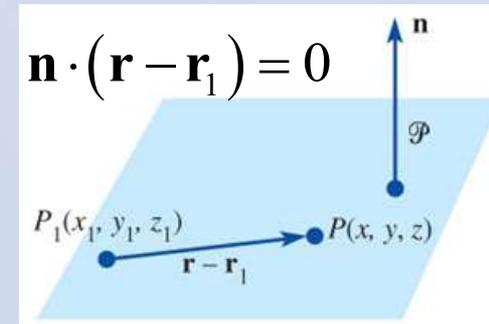
# Example of Planes in 3-Space

1. จงหาสมการของระนาบ (vector equation for a plane) ซึ่งมีเวกเตอร์ตั้งฉาก  $\mathbf{n} = 2\mathbf{i} + 8\mathbf{j} - 5\mathbf{k}$  และมีจุด  $(4, -1, 3)$  อยู่บนระนาบดังกล่าว?

$$\overrightarrow{P_1P} = \mathbf{r} - \mathbf{r}_1 = (x - 4)\mathbf{i} + (y + 1)\mathbf{j} - (z - 3)\mathbf{k} = 0$$

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_1) = 2(x - 4) + 8(y + 1) - 5(z - 3) = 0$$

$$\therefore \text{สมการของระนาบ คือ } 2x + 8y - 5z + 15 = 0$$



2. จงหาสมการของระนาบที่มีจุด  $(1, 0, -1)$ ,  $(3, 1, 4)$  และ  $(2, -2, 0)$  อยู่บนระนาบ

$$\left. \begin{matrix} (3, 1, 4) \\ (1, 0, -1) \end{matrix} \right\} \mathbf{u} = 2\mathbf{i} + 1\mathbf{j} + 5\mathbf{k},$$

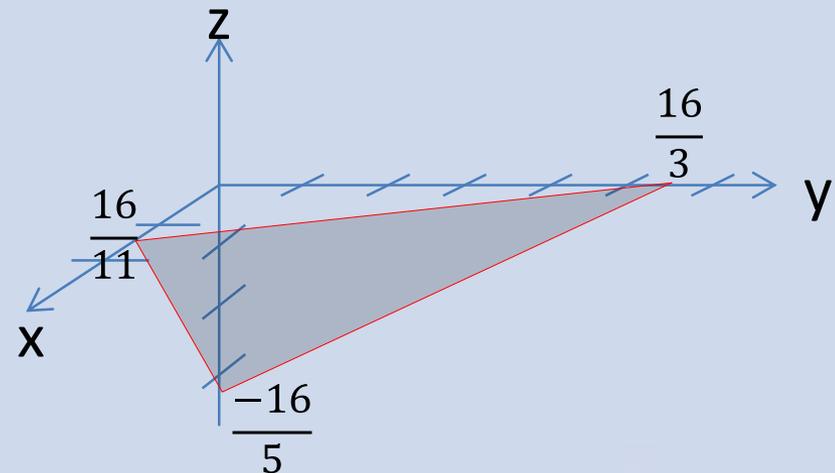
$$\left. \begin{matrix} (3, 1, 4) \\ (2, -2, 0) \end{matrix} \right\} \mathbf{v} = 1\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$$

$$\left. \begin{matrix} (x, y, z) \\ (2, -2, 0) \end{matrix} \right\} \mathbf{w} = (x - 2)\mathbf{i} + (y + 2)\mathbf{j} + z\mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 5 \\ 1 & 3 & 4 \end{vmatrix} = -11\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$$

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = 0 = -11(x - 2) - 3(y + 2) + 5z$$

$$-11x - 3y + 5z + 16 = 0$$

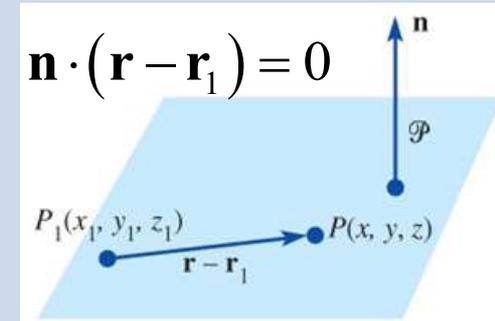


# Example of Planes in 3-Space

3. จงหาสมการของระนาบที่มีจุด  $(4,3,1)$ ,  $(6,4,4)$  และ  $(1,2,-1)$  อยู่บนระนาบ

$$\begin{array}{l} (1,2,-1) \\ (4,3,1) \end{array} \left. \vphantom{\begin{array}{l} (1,2,-1) \\ (4,3,1) \end{array}} \right\} u=3i+1j+2k$$
$$\begin{array}{l} (4,3,1) \\ (6,4,4) \end{array} \left. \vphantom{\begin{array}{l} (4,3,1) \\ (6,4,4) \end{array}} \right\} v=2i+1j+3k$$
$$\begin{array}{l} (4,3,1) \\ (x,y,z) \end{array} \left. \vphantom{\begin{array}{l} (4,3,1) \\ (x,y,z) \end{array}} \right\} w=(x-4)i+(y-3)j+(z-1)k$$

$$u \times v = \begin{vmatrix} i & j & k \\ 3 & 1 & 2 \\ 2 & 1 & 3 \end{vmatrix} = i - 5j + k$$



$$n \cdot (r - r_1) = (u \times v) \cdot w = (x-4) - 5(y-3) + (z-1) = 0$$

$$x - 5y + z + 10 = 0$$

# Examples

## EXAMPLE 10 Graph of a Plane

Graph the equation  $2x + 3y + 6z = 18$ .

## EXAMPLE 11 Graph of a Plane

Graph the equation  $6x + 4y = 12$ .

## EXAMPLE 12 Graph of a Plane

Graph the equation  $x + y - z = 0$ .

# Orthonormal Basis (ฐานเชิงตั้งฉาก)

- Every vector  $\mathbf{u}$  in  $R^2$  ( $n=2$ ) can be written as a linear combination of the vectors in the standard basis  $B = \{\mathbf{e}_1, \mathbf{e}_2\} = \{\mathbf{i}, \mathbf{j}\}$ , where  $\mathbf{i} = \langle 1, 0 \rangle$ ,  $\mathbf{j} = \langle 0, 1 \rangle$
- Every vector  $\mathbf{u}$  in  $R^3$  can be written as a linear combination of the vectors in the standard basis  $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ , where  $\mathbf{i} = \langle 1, 0, 0 \rangle$ ,  $\mathbf{j} = \langle 0, 1, 0 \rangle$ ,  $\mathbf{k} = \langle 0, 0, 1 \rangle$

# Orthonormal Basis (ฐานเชิงตั้งฉาก)

- Orthonormal basis = mutually orthogonal ( $\mathbf{e}_i \cdot \mathbf{e}_j = 0, i \neq j$ ) and unit vectors ( $\|\mathbf{e}_i\| = 1$ )
- HOW TO transform or convert any basis  $B$  of  $R^2$  or  $R^3$  into an orthonormal basis?

# Example of Orthonormal Basis for $R^3$

The set of three vectors

$$\mathbf{w}_1 = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle, \mathbf{w}_2 = \left\langle -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle, \mathbf{w}_3 = \left\langle 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle \quad (1)$$

is linearly independent and spans the space  $R^3$ . Hence  $B = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$  is a basis for  $R^3$ . Using the standard inner product or dot product defined on  $R^3$ , observe

$$\mathbf{w}_1 \cdot \mathbf{w}_2 = 0, \mathbf{w}_1 \cdot \mathbf{w}_3 = 0, \mathbf{w}_2 \cdot \mathbf{w}_3 = 0, \quad \text{and} \quad \|\mathbf{w}_1\| = 1, \|\mathbf{w}_2\| = 1, \|\mathbf{w}_3\| = 1.$$

Hence  $B$  is an orthonormal basis. ≡

A basis  $B$  for  $R^n$  need not be orthogonal nor do the basis vectors need to be unit vectors.

$$\mathbf{u}_1 = \langle 1, 0, 0 \rangle, \quad \mathbf{u}_2 = \langle 1, 1, 0 \rangle, \quad \mathbf{u}_3 = \langle 1, 1, 1 \rangle$$

in  $R^3$  are linearly independent and hence  $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$  is a basis for  $R^3$ . Note that  $B$  is not an orthogonal basis.

Generally, an orthonormal basis for a vector space  $V$  turns out to be the most convenient basis for  $V$ . One of the advantages that an orthonormal basis has over any other basis for  $R^n$  is the comparative ease with which we can obtain the coordinates of a vector  $\mathbf{u}$  relative to that basis.

# Orthonormal Basis

## Theorem 7.7.1 Coordinates Relative to an Orthonormal Basis

Suppose  $B = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$  is an orthonormal basis for  $R^n$ . If  $\mathbf{u}$  is any vector in  $R^n$ , then

$$\mathbf{u} = (\mathbf{u} \cdot \mathbf{w}_1)\mathbf{w}_1 + (\mathbf{u} \cdot \mathbf{w}_2)\mathbf{w}_2 + \cdots + (\mathbf{u} \cdot \mathbf{w}_n)\mathbf{w}_n.$$

$$\mathbf{w}_1 = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle, \mathbf{w}_2 = \left\langle -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle, \mathbf{w}_3 = \left\langle 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$$

Find the coordinates of the vector  $\mathbf{u} = \langle 3, -2, 9 \rangle$  relative to the orthonormal basis  $B$  for  $R^3$  given in (1) of Example 1. Write  $\mathbf{u}$  in terms of the basis  $B$ .

**SOLUTION** From Theorem 7.7.1, the coordinates of  $\mathbf{u}$  relative to the basis  $B$  in (1) of Example 1 are simply

$$\mathbf{u} \cdot \mathbf{w}_1 = \frac{10}{\sqrt{3}}, \quad \mathbf{u} \cdot \mathbf{w}_2 = \frac{1}{\sqrt{6}}, \quad \text{and} \quad \mathbf{u} \cdot \mathbf{w}_3 = -\frac{11}{\sqrt{2}}.$$

Hence we can write

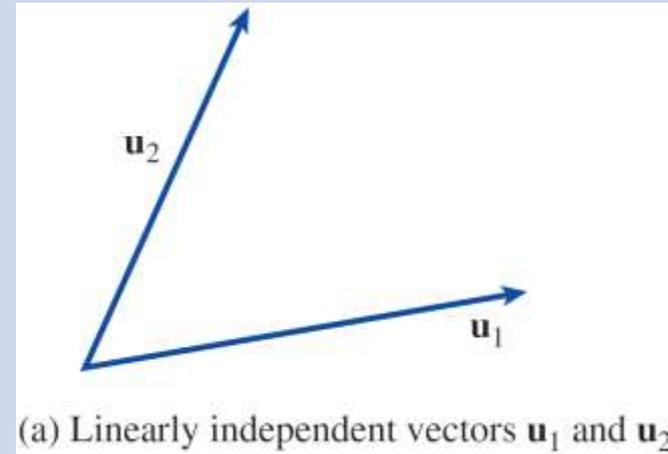
$$\mathbf{u} = \frac{10}{\sqrt{3}}\mathbf{w}_1 + \frac{1}{\sqrt{6}}\mathbf{w}_2 - \frac{11}{\sqrt{2}}\mathbf{w}_3.$$

# Gram–Schmidt Orthogonalization Process (แกรม-ชมิทธ์ ออโธโกนอลไลเซชันโพรเซส)

- Gram–Schmidt orthogonalization process is an algorithm for generating an orthogonal basis  $B' = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ , from any given basis  $B = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ , for  $R^n$
- How?
- Key idea in the orthogonalization process is “vector projection”
- (Review:  $\text{proj}_b \mathbf{a}$ )
- Creating an orthonormal basis  $B'' = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$  by normalizing the vectors in the orthogonal basis  $B'$

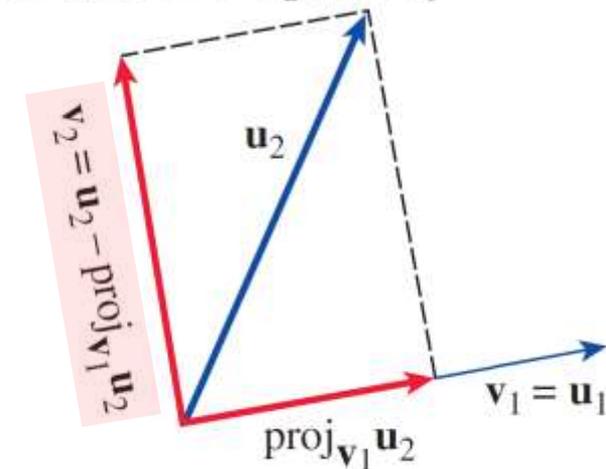
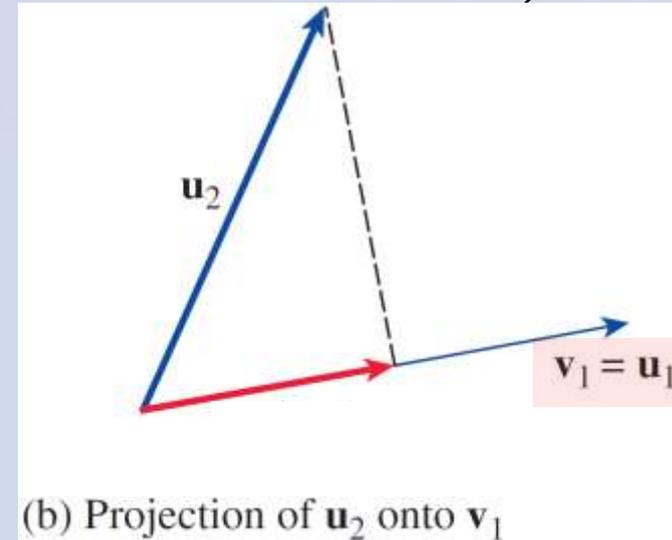
# Gram–Schmidt Orthogonalization Process (Constructing an Orthogonal Basis for $\mathbb{R}^2$ )

- Transformation of a basis  $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ , for  $\mathbb{R}^2$  into an orthogonal basis  $B' = \{\mathbf{v}_1, \mathbf{v}_2\}$  consists of 2 steps.



# Gram–Schmidt Orthogonalization Process (Constructing an Orthogonal Basis for $\mathbb{R}^2$ )

- 1<sup>st</sup> step, we choose one of the vectors in  $B$ , say,  $\mathbf{u}_1$ , and rename it as  $\mathbf{v}_1$
- Next, we project the remaining vector  $\mathbf{u}_2$  in  $B$  onto the vector  $\mathbf{v}_1$  ( $\text{proj}_{\mathbf{v}_1} \mathbf{u}_2$ )
- ,and define a 2<sup>nd</sup> vector to be  $\mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_{\mathbf{v}_1} \mathbf{u}_2$
- $\text{proj}_{\mathbf{v}_1} \mathbf{u}_2 = \left( \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1$

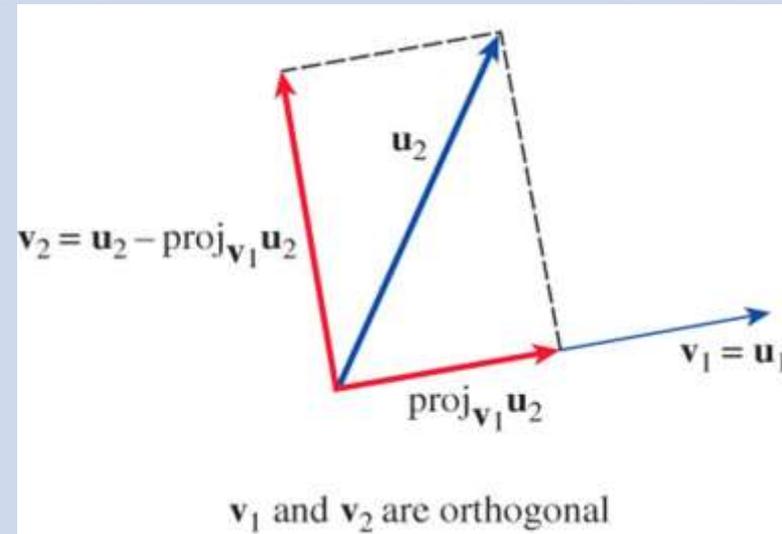


# Gram–Schmidt Orthogonalization Process (Constructing an Orthogonal Basis for $\mathbb{R}^2$ )

$$\mathbf{v}_1 = \mathbf{u}_1$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \left( \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1$$

$$\mathbf{v}_2 \cdot \mathbf{v}_1 = 0$$



# Gram–Schmidt Orthogonalization Process (Constructing an Orthogonal Basis for $\mathbb{R}^2$ )

The set  $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ , where  $\mathbf{u}_1 = \langle 3, 1 \rangle$ ,  $\mathbf{u}_2 = \langle 1, 1 \rangle$ , is a basis for  $\mathbb{R}^2$ . Transform  $B$  into an orthonormal basis  $B'' = \{\mathbf{w}_1, \mathbf{w}_2\}$ .

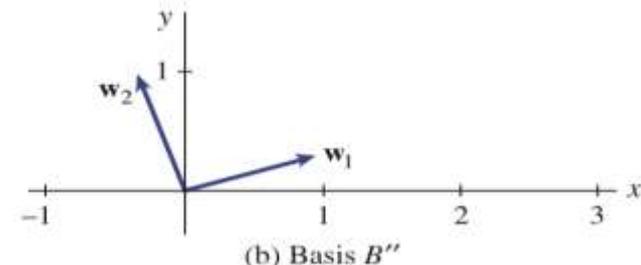
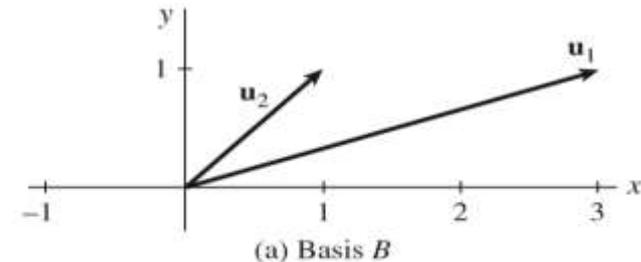
**SOLUTION** We choose  $\mathbf{v}_1$  as  $\mathbf{u}_1$ :  $\mathbf{v}_1 = \langle 3, 1 \rangle$ . Then from the second equation in (3), with  $\mathbf{u}_2 \cdot \mathbf{v}_1 = 4$  and  $\mathbf{v}_1 \cdot \mathbf{v}_1 = 10$ , we obtain

$$\mathbf{v}_2 = \langle 1, 1 \rangle - \frac{4}{10} \langle 3, 1 \rangle = \left\langle -\frac{1}{5}, \frac{3}{5} \right\rangle.$$

The set  $B' = \{\mathbf{v}_1, \mathbf{v}_2\} = \{\langle 3, 1 \rangle, \langle -\frac{1}{5}, \frac{3}{5} \rangle\}$  is an orthogonal basis for  $\mathbb{R}^2$ . We finish by normalizing the vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$ :

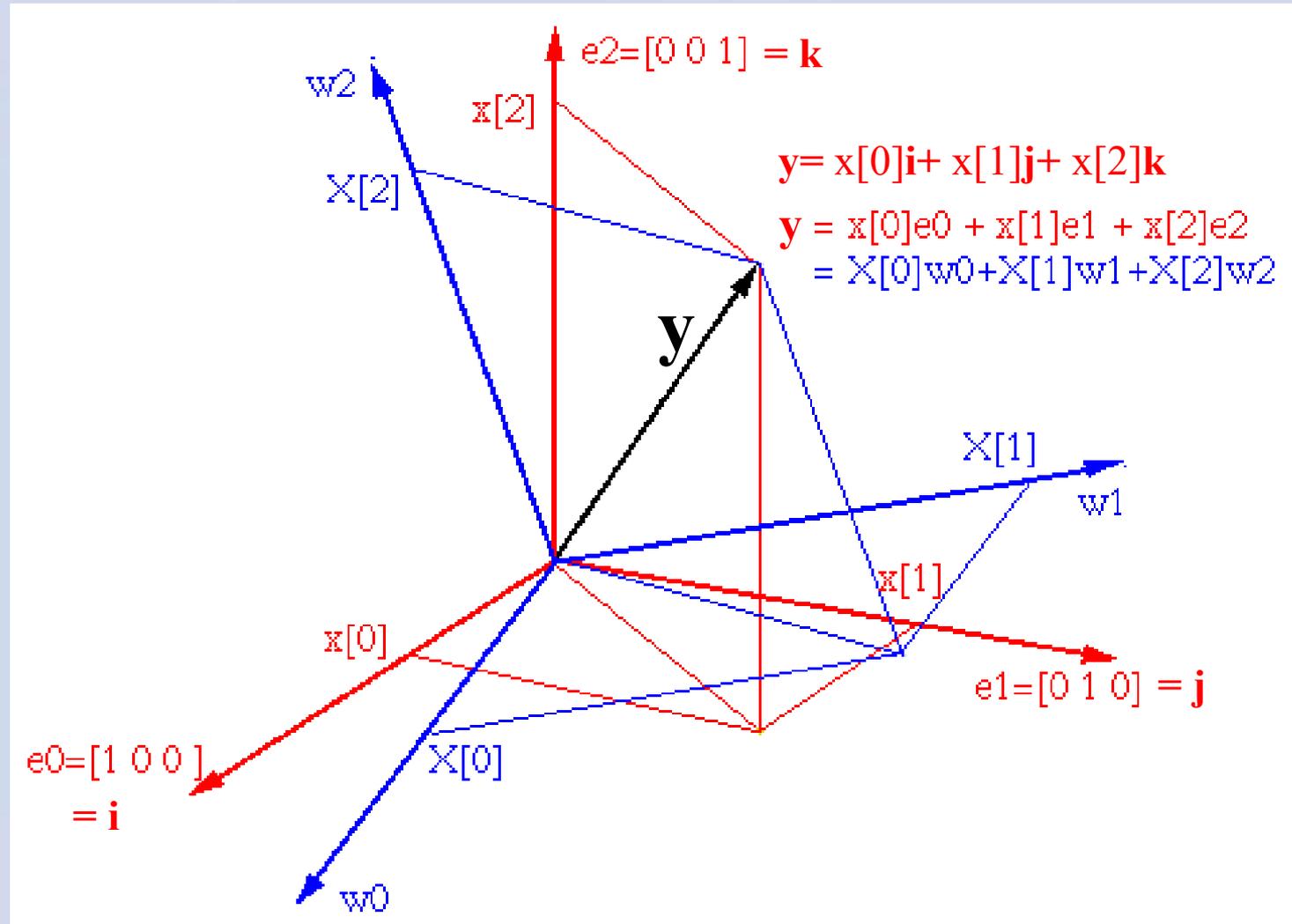
$$\mathbf{w}_1 = \frac{1}{\|\mathbf{v}_1\|} \mathbf{v}_1 = \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle \quad \text{and} \quad \mathbf{w}_2 = \frac{1}{\|\mathbf{v}_2\|} \mathbf{v}_2 = \left\langle -\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle.$$

The basis  $B$  is shown in **FIGURE 7.7.2(a)**, and the new orthonormal basis  $B'' = \{\mathbf{w}_1, \mathbf{w}_2\}$  is shown in blue in Figure 7.7.2(b).



In Example above we are free to choose either vector in  $B = \{\mathbf{u}_1, \mathbf{u}_2\}$  as the vector  $\mathbf{v}_1$ . However, by choosing  $\mathbf{v}_1 = \mathbf{u}_2 = \langle 1, 1 \rangle$ , we obtain a different orthonormal basis, namely,  $B'' = \{\mathbf{w}_1, \mathbf{w}_2\}$ , where  $\mathbf{w}_1 = \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle$  and  $\mathbf{w}_2 = \langle 1/\sqrt{2}, -1/\sqrt{2} \rangle$ .

# Example of Orthonormal Basis for $R^3$



# Gram–Schmidt Orthogonalization Process (Constructing an Orthogonal Basis for $\mathbb{R}^3$ )

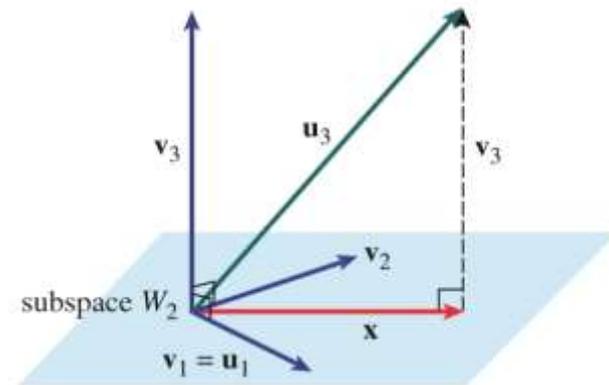
Now suppose  $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$  is a basis for  $\mathbb{R}^3$ .

Then the set  $B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ , where

$$\mathbf{v}_1 = \mathbf{u}_1$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \left( \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1$$

$$\mathbf{v}_3 = \mathbf{u}_3 - \left( \frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 - \left( \frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \right) \mathbf{v}_2$$



# Gram–Schmidt Orthogonalization Process (Constructing an Orthogonal Basis for $\mathbb{R}^3$ )

The set  $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ , where  $\mathbf{u}_1 = \langle 1, 1, 1 \rangle$ ,  $\mathbf{u}_2 = \langle 1, 2, 2 \rangle$ ,  $\mathbf{u}_3 = \langle 1, 1, 0 \rangle$  is a basis for  $\mathbb{R}^3$ . Transform  $B$  into an orthonormal basis  $B''$ .

**SOLUTION** We choose  $\mathbf{v}_1$  as  $\mathbf{u}_1$ :  $\mathbf{v}_1 = \langle 1, 1, 1 \rangle$ . Then from the second equation in (4), with  $\mathbf{u}_2 \cdot \mathbf{v}_1 = 5$  and  $\mathbf{v}_1 \cdot \mathbf{v}_1 = 3$ , we obtain

$$\mathbf{v}_2 = \langle 1, 2, 2 \rangle - \frac{5}{3} \langle 1, 1, 1 \rangle = \left\langle -\frac{2}{3}, \frac{1}{3}, \frac{1}{3} \right\rangle.$$

Now with  $\mathbf{u}_3 \cdot \mathbf{v}_1 = 2$ ,  $\mathbf{v}_1 \cdot \mathbf{v}_1 = 3$ ,  $\mathbf{u}_3 \cdot \mathbf{v}_2 = -\frac{1}{3}$ , and  $\mathbf{v}_2 \cdot \mathbf{v}_2 = \frac{2}{3}$ , the third equation in (4) yields

$$\begin{aligned} \mathbf{v}_3 &= \langle 1, 1, 0 \rangle - \frac{2}{3} \langle 1, 1, 1 \rangle + \frac{1}{2} \left\langle -\frac{2}{3}, \frac{1}{3}, \frac{1}{3} \right\rangle \\ &= \langle 1, 1, 0 \rangle + \left\langle -\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3} \right\rangle + \left\langle -\frac{1}{3}, \frac{1}{6}, \frac{1}{6} \right\rangle = \left\langle 0, \frac{1}{2}, -\frac{1}{2} \right\rangle. \end{aligned}$$

The set  $B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{\langle 1, 1, 1 \rangle, \langle -\frac{2}{3}, \frac{1}{3}, \frac{1}{3} \rangle, \langle 0, \frac{1}{2}, -\frac{1}{2} \rangle\}$  is an orthogonal basis for  $\mathbb{R}^3$ .

$B'' = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ , where

$$A = [1 \ 1 \ 1; 1 \ 2 \ 2; 1 \ 1 \ 0];$$

$B = \text{grams}(A)$

$$\mathbf{w}_1 = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle, \quad \mathbf{w}_2 = \left\langle -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle, \quad \mathbf{w}_3 = \left\langle 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle.$$

# Gram–Schmidt Orthogonalization Process

## Theorem 7.7.2 Gram–Schmidt Orthogonalization Process

Let  $B = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ ,  $m \leq n$ , be a basis for a subspace  $W_m$  of  $R^n$ . Then  $B' = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ , where

$$\begin{aligned}\mathbf{v}_1 &= \mathbf{u}_1 \\ \mathbf{v}_2 &= \mathbf{u}_2 - \left( \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 \\ \mathbf{v}_3 &= \mathbf{u}_3 - \left( \frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 - \left( \frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \right) \mathbf{v}_2 \\ &\vdots \\ \mathbf{v}_m &= \mathbf{u}_m - \left( \frac{\mathbf{u}_m \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 - \left( \frac{\mathbf{u}_m \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \right) \mathbf{v}_2 - \dots - \left( \frac{\mathbf{u}_m \cdot \mathbf{v}_{m-1}}{\mathbf{v}_{m-1} \cdot \mathbf{v}_{m-1}} \right) \mathbf{v}_{m-1},\end{aligned}\tag{7}$$

is an orthogonal basis for  $W_m$ . An orthonormal basis for  $W_m$  is

$$B'' = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\} = \left\{ \frac{1}{\|\mathbf{v}_1\|} \mathbf{v}_1, \frac{1}{\|\mathbf{v}_2\|} \mathbf{v}_2, \dots, \frac{1}{\|\mathbf{v}_m\|} \mathbf{v}_m \right\}.$$

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- สมมุติว่า  $B = \langle \mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n \rangle$  เป็นฐานตั้งฉาก (Orthogonal basis) สำหรับ  $\mathbf{R}^n$  และถ้า  $\mathbf{u}$  เป็นเวกเตอร์ใดๆใน  $\mathbf{R}^n$ , ดังนั้น
$$\mathbf{u} = (\mathbf{u} \cdot \mathbf{w}_1)\mathbf{w}_1 + (\mathbf{u} \cdot \mathbf{w}_2)\mathbf{w}_2 + \dots + (\mathbf{u} \cdot \mathbf{w}_n)\mathbf{w}_n$$
- ฐาน  $B$  ของ  $\mathbf{R}^n$  สามารถถูกแปลงเป็นฐานที่ตั้งฉาก  $B' = \langle \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \rangle$  แล้วแปลงเป็นฐานที่ตั้งฉาก  $B'' = \langle \mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n \rangle$  โดยการทำนอร์มอลไลซ์เวกเตอร์ใน  $B'$
- เวกเตอร์  $\mathbf{v}_n$  และ  $\mathbf{w}_n$  เป็นเวกเตอร์ที่ตั้งฉากกันและเป็นเวกเตอร์หน่วย

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- ตัวอย่าง: การแปลงเซต  $B = \{\mathbf{u}_1, \mathbf{u}_2\}$  ไปเป็นฐานตั้งฉาก  $B'' = \{\mathbf{w}_1, \mathbf{w}_2\}$  (โดยที่  $\mathbf{u}_1 = \langle 3, 1 \rangle$  และ  $\mathbf{u}_2 = \langle 1, 1 \rangle$ )
  - เลือก  $\mathbf{v}_1 = \mathbf{u}_1 = \langle 3, 1 \rangle$  และ  $\mathbf{v}_2 = \mathbf{u}_2 - \left( \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 = \langle 1, 1 \rangle - \frac{4}{10} \langle 3, 1 \rangle = \left\langle -\frac{1}{5}, \frac{3}{5} \right\rangle$
  - เซต  $B' = \left\{ \langle 3, 1 \rangle, \left\langle -\frac{1}{5}, \frac{3}{5} \right\rangle \right\}$  จะมีฐานตั้งฉากสำหรับ  $R^2$

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- ตัวอย่าง:

- ทำได้โดยการนอร์มัลไลซ์เวกเตอร์  $\mathbf{v}_1$  และ  $\mathbf{v}_2$

$$\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle \text{ และ } \mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \left\langle -\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle$$

- ฐานใหม่ที่ตั้งฉาก (new orthonormal basis ) เท่ากับ

- $B'' = \{\mathbf{w}_1, \mathbf{w}_2\} = \left\{ \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle, \left\langle -\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle \right\}$